Max Learning's Algebra Antics

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Welcome!

Hi, my name is Max Learning, and I'll be your teacher and guide.

My goal is to make math seem "real" to you, so you'll gain confidence and *look forward* to your next math challenge.

The fact that you're reading this book means you're eager to succeed and are willing to explore new ways to do so. *So let's get started!*

Why Is Math A Struggle?	How This Book Can Help
Symbols Math uses symbols, *lots* of them. It's as difficult to learn as a foreign language.	**Mental Manipulatives** You'll learn to "see" three-dimensional objects behind each symbol.
Rules Math is based on rules, *lots* of them. It's hard not to confuse one for the other.	**BrainAids** You'll learn clever memory hints that make the rules easy and fun.
Trauma Getting an answer wrong in front of the class, losing at a flash-card competition, failing a test, being criticized by a teacher—all can lead to math trauma.	**RUFF** You'll learn to be in a Relaxed, Uncluttered, Focused, and Flowing state of mind, which increases confidence and eases past traumas.

What's Good About Math?

Certainty
Math problems have *right* answers. In most subjects, like English or Art, the grade you get on an essay or project depends on your teacher's opinion of your work. However, in a Math class, when you get the right answer, no one can argue with it. It's certain!

Quest
Math problems are puzzles. The quest to solve them can be exciting! If you approach it with this attitude, math can be as fun and engaging as any game you'll ever play. Solving problems that others find difficult is very satisfying and makes you feel smart!

Magic
Math is the *language of nature*. Nearly everything we see, hear, touch and do can be described with math. Every image, color, and sound on a computer is based on math. In today's movies, you can't always tell what's real and what's been generated by some mathematical formula. In short, math is amazing—there's magic in it!

Note to Parents

I've kept the problems in this book simple, so you and your kids could grasp the concepts without getting bogged down in the arithmetic. And I've tried to make it as interesting and memorable as possible with illustrations, Mental Manipulatives, and BrainAids.

But don't be surprised if your kids don't rush to do math on their own. Except for the rare few who find it fun and challenging, most avoid math like the plague. After all, it's not always easy, and most of us avoid uncomfortable mental effort whenever possible.

But math is a school requirement, students have to learn it, so I try to make it as painless as possible. And many children, once they "see the light" and have tasted success, come to enjoy the subject.

If your child is not motivated to read this book, or has trouble understanding some of the concepts or techniques, I recommend you first learn them on your own, then teach them to your child. It's what I would do in a classroom or tutoring session. I only wrote the book because I can't be everywhere to teach every student. Besides, most of us would rather be shown how to do something rather than having to read about it.

This is a *techniques* book rather than a *drill & practice* book. Check your answers to the **Your turn** activities in the **Answer Key** in the back of the book. Then apply these techniques to the numerous problems in traditional math textbooks, or try to make up some problems of your own.

You're learning a new, I hope, more interesting way of doing math. As with learning anything new, it's best not to rush; so relax, take your time, and enjoy the process!

Pronunciation Guide

Sometimes it may not be obvious how to pronounce terms you have not heard spoken. When you see a term followed by a pronunciation, refer to this guide as needed.

Vowels		
Long	**Short**	**Other**
aa = ate	a = act	ai=air, ar=are, aw=paw
ee = eel	e/eh = end	
ii = hi	i/ih = hid	
oh = no	aw = on	oo = book, or = for ow = how, oy = boy
yu = use	u/uh = up	uu = too, ur = fur

Consonants	
Hard	**Soft**
k = cat	s = ice
g = go	j = gem
s/ss = hiss	z = his
ch = chin	sh=shin; zh=vision
th = thin	thh = this
Accent on: UP-ur-KAASS	

Common Abbreviations

aka = also known as

e.g. = for example (think egzample)

i.e. = that is

p. = page

FYI = For Your Information

BrainAids

It was a mouthful to say *mnemonic* (nee-MAWN-ik) *device,* so I coined the word *BrainAid* for memory hints that help you learn. Feel free to make up your own BrainAids. Most fall into one of the following "A" categories:

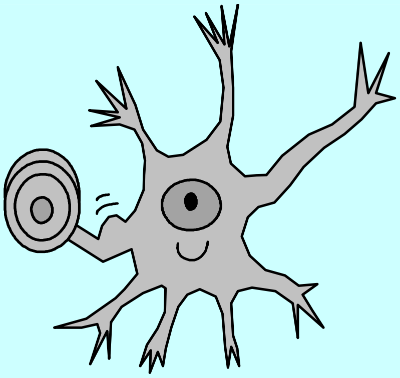
Analogy = Comparison

How to say it: uh-NOWL-uh-jee

What it is: A *comparison* of what you are trying to learn to what you already know.

Why it works: To learn, new brain fibers must grow and laboriously push their way through dense brain tissue, which can be very tiring. An analogy lets you piggyback new knowledge on top of *existing* brain fibers, which is quicker and takes much less effort.

Analogy Example: Just as *physical* exercise builds new *muscle* fibers, *mental* exercise builds new *brain* fibers. Both take time, effort, and repetition.



Analogy: Building brain fibers.

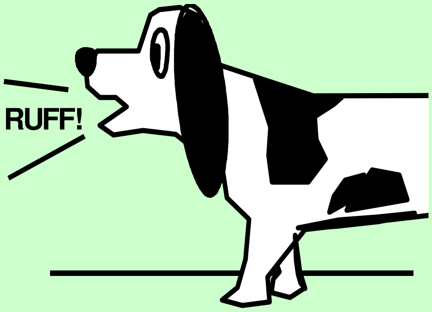
Acronym = Name

How to say it: AK-roh-nim

What it is: A *name* made from the first letters of several words. Hint: Think *nym* = *name*.

Why it works: The letters of an acronym act like small hooks on which you can hang large words. You memorize a little to remember a lot.

Acronym Example: To maximize your learning, be in a learning frame of mind: Relaxed, Uncluttered, Focused, and Flowing. In other words, be RUFF.



Acronym: RUFF

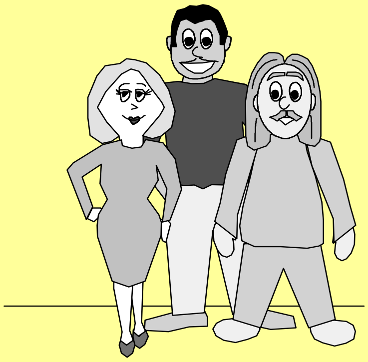
Acrostic = Story

How to say it: uh-KRAW-stik

What it is: A *story* made from the first letters of several words. Hint: Think *stic* = *st*ory.

Why it works: Sometimes a "story" is easier to remember than an acronym, especially if the acronym is hard to pronounce.

Acrostic Example: You create the acronym MTF to help you remember the names of three new acquaintances, Mary, Tom, and Fred. But MTF is hard to pronounce, so you convert it into the acrostic "My Three Friends."



Acrostic: My Three Friends

Concepts

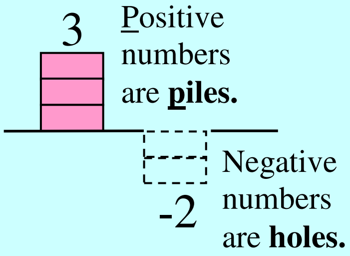
Math Basics

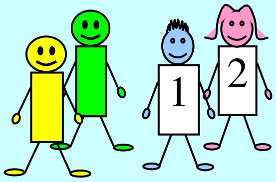
In *Max Learning's Mental Math* and *Fraction Fun* books,
we learned several concepts that will help us in *Algebra Antics.*
Please refer to these books for more details on the following Math Basics concepts.

Mental Manipulatives

Traditional manipulatives are physical objects, like tiles or blocks, which you "manipulate" to mimic math operations. *Mental* manipulatives are items you visualize when you see a number or operation. They can turn lifeless symbols into reality—at least in your imagination.

And unlike physical manipulatives, mental manipulatives are always with you to make math more engaging. Mental Manipulatives include piles, holes, MathBots, and many other items.





MathBots manipulate piles and holes or represent numbers.

Numbers

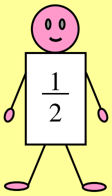
A number is a symbol for a quantity or value.

Natural Numbers: Counting numbers: 1, 2, 3... **+**

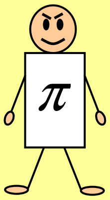
Whole Numbers: Zero + Natural numbers: 0, 1, 2, 3... **0+**

Integers: Negatives of Natural numbers + Whole numbers: ...-3, -2, -1, 0, 1, 2, 3... **−0+**

Rational Numbers: Can be written as *ratio*s.
Consist of integers, fractions, terminating or repeating decimals: 2, ½, .33

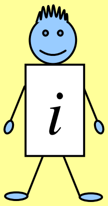


Irrational Numbers: Can *not* be written as ratios.
Consist of non-repeating or non-terminating decimals: π, $\sqrt{2}$



Real Numbers: All rational and irrational numbers.

Imaginary Number: $\sqrt{-1}$ or *i*



Complex Numbers: Real number with imaginary number: 3 + *i*

Operators & Operands

Operand **operator** ***Operand***

An operator is a symbol for a procedure or relationship between operands.
Operands include: addends, minuends & subtrahends, multipliers & multiplicands, dividends & divisors.

Arithmetic Operators

Arithmetic operators specify procedures.

+	Add
–	Subtract
× •	Multiply
÷ /	Divide
±	Plus or Minus

Relational Operators

Relational operators specify relationships.

=	Equal
≠	Not equal to
>	Greater than
<	Less than
≥	Greater than or equal to
≤	Less than or equal to

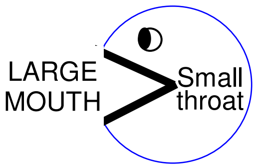
Computer Operators

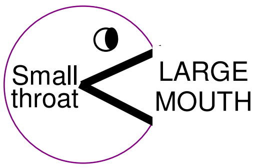
Many of the common operators do *not* appear on computer keyboards. Below are alternates, typically used in computer spreadsheet formulas.

*	Asterisk (aka star) for multiply
^	Caret [KAIR-et] for exponentiation.
<>	Not equal to
>=	Greater than or equal to
<=	Less than or equal to

BrainAid

Be careful not to confuse the > and < symbols.
The *larger* number goes on the *larger* side.
Example: 7 > 6; 6 < 7





Algorithms

An algorithm [AL-goh-RITHH-um] is a *step-by-step procedure*.

Algorithms make math operations easier.

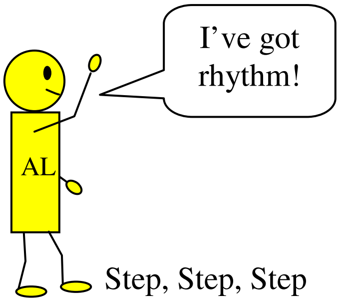
Instead of having to figure out what to do each time, you follow the algorithm.

For example, the procedure you follow when doing long division is an algorithm.

BrainAid

Al go(t) rithm.

He smartly follows the dance’s *step-by-step procedure*.



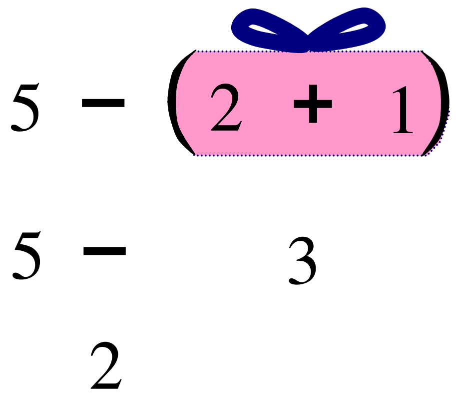
PEMDAS

Priority of Operations

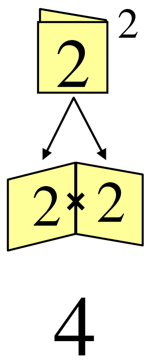
When a math problem has more than one operator, work in this order:

- Parentheses: Perform operations inside of parentheses first.
  If nested, start with the innermost set of parentheses: (Do 2nd (do 1st)).
- Exponentiation: Raise numbers to powers.
- Multiplication/Division: If encounter both, perform in left-to-right order.
- Addition/Subtraction: If encounter both, perform in left-to-right order.

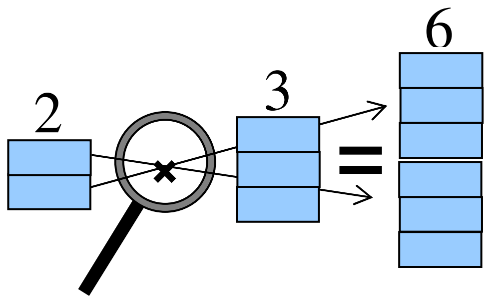
Parentheses Package



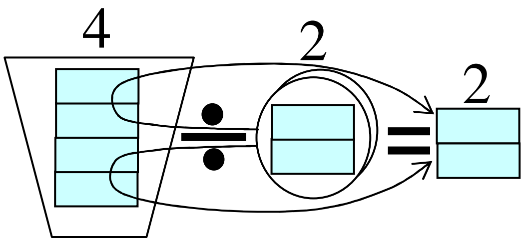
Exponentiation Expands



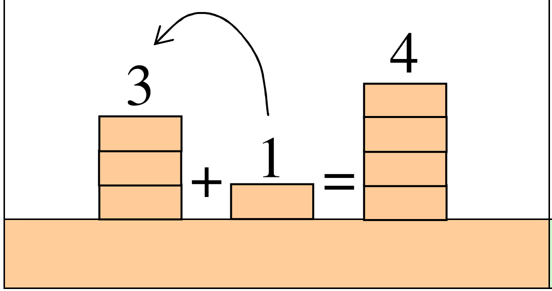
Multiplication Magnifies



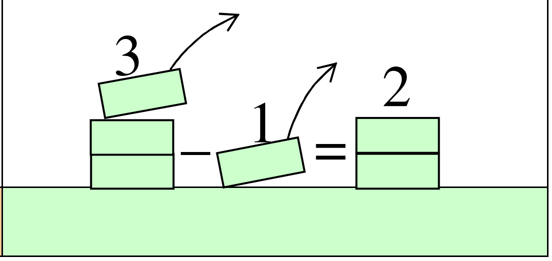
Division Dissolves



Addition Attaches



Subtraction Steals



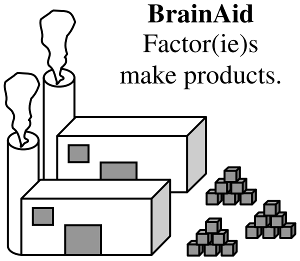
Factors

Factors are *multipliers* that combine to make products.

Factor × Factor = Product
Example: 2 × 3 = 6, so 2 and 3 are factors of the product 6.

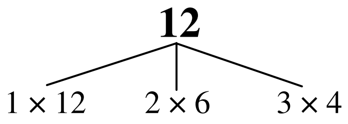
Factoring is the process of finding a product's factors.

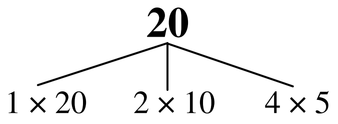
To factor means to extract the multipliers that form a product.
Example: 6 can be factored into 1 x 6 or 2 x 3, so the factors of 6 are 1, 2, 3, 6.



BrainAid
Factor(ie)s
make products.

Common Factors are factors that are the same for different products.





	Factors					
Product 12	1	2	3	4	6	12
Product 20	1	2	4	5	10	20

GCF

1, 2, and 4 are common factors of the products 12 and 20.
4 is the Greatest Common Factor (GCF) of 12 and 20.

Why Factor?
One reason is to make numbers smaller and easier to work with, e.g., reducing fractions to their lowest terms.

Factoring Tricks

Use these tricks to see if a number contains a factor *before* you waste time trying to extract it.

A product is evenly* divisible by a factor of:

2—If the product is even (i.e., ends in 0, 2, 4, 6, or 8).

3—If the sum of the product's digits is a multiple of 3 (321: 3+2+1 = 6).

4—If the product's last 2 digits are a multiple of 4 (316).

5—If the product ends in 0 or 5 (765).

6—If the product fits the tricks for both 2 and 3 above (462: 4+6+2 = 12).

7—If the product's 1st digits minus (2 × the last digit) is 0 or multiple of 7 [112: 11–(2×2) = 11 – 4 = 7].

8—If the product's last 3 digits are 000 or a multiple of 8 (2104).

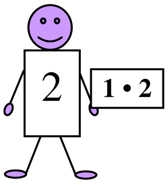
9—If sum of the product's digits is a multiple of 9 (864: 8+6+4 = 18).

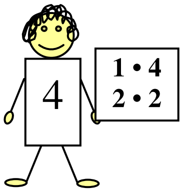
* Technically, *every* number is divisible by *every* number (except 0), but may not be evenly so; e.g., 10÷4 = 2½

Composite factors are divisible by 1, themselves, and at least one other number.
Example: 4 is divisible by 1 and 4, but also by 2.

Prime factors are divisible by 1 and themselves only.
Example: 2 is divisible by 1 and 2 only. The same is true for 3, 5, 7, 11, etc.

0 and 1 by definition are neither composite nor prime.





Tip: To ensure complete factoring, factor until all factors are prime numbers.

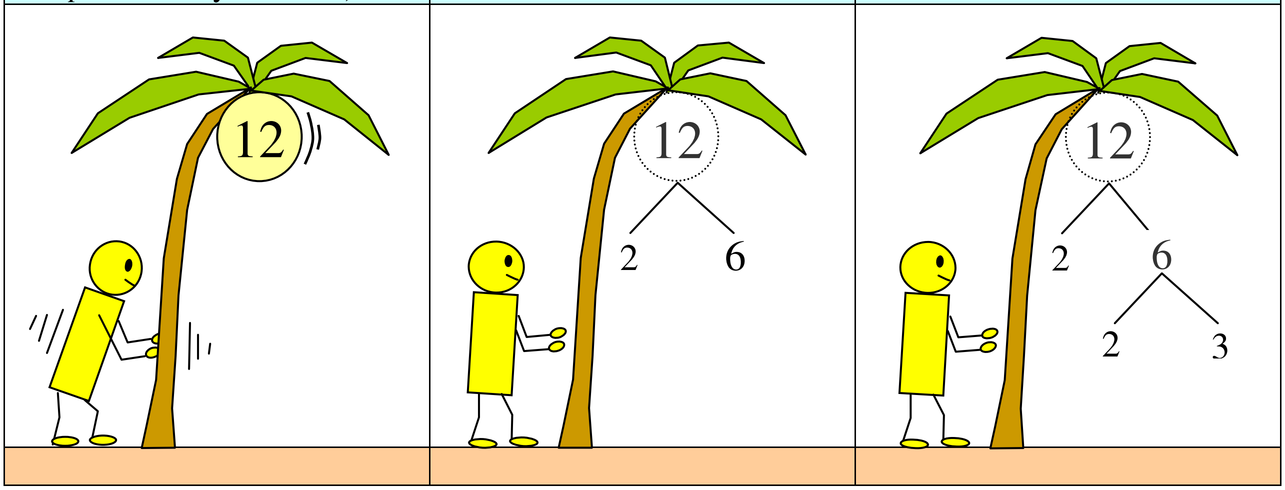
Factor Trees

Factor Trees are useful for extracting prime factors.

Tropical Factor Tree

Imagine being on an island with a palm tree containing a coconut. Being hungry, you grip and shake the tree. As the coconut falls, it conveniently splits in smaller pieces full of prime nutrients for you to eat.

Grip and shake the tree to dislodge the coconut (the *product* of your labor).	As it falls, the coconut splits into smaller pieces (*factors*).	On the way down, all pieces split into their *prime* nutrients.

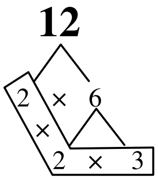


Traditional Factor Tree

To create a Factor Tree without having to call on your artistic ability:

- Draw two branches beneath the product to be factored.
- Extract the smallest prime factor (2, 3, 5, etc.) and place it under the left branch with the composite factor under the right branch.
- Repeat the process with the composite factor until all factors are prime.
- Box the prime numbers at the bottom of the branches.

Factor Tree



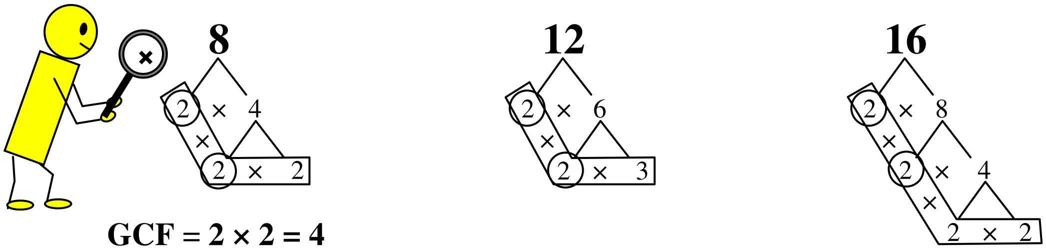
GCF: <u>G</u>rip, <u>C</u>atch, <u>F</u>ocus

To find the GCF of several products:

- <u>G</u>rip each products' Factor Tree, and shake out its prime factors.
- <u>C</u>atch (circle) factors that are common to all products each time they occur.
- <u>F</u>ocus on and magnify (multiply) *one* set of circled factors to get the GCF.

GCF Paradox
The *Greatest* Common Factor is *less* than the products it's derived from.

Example: Find the GCF of 8, 12, and 16.



GCF = 2 × 2 = 4

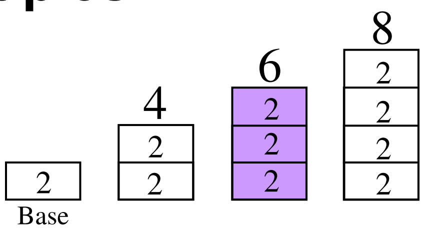
Observe that there are two 2s that are common to all products, so they are both circled.
Observe that only *one* set of common factors is multiplied to find the GCF.
Example of use: Extracting the GCF of 4 from 8, 12, and 16 reduces them to 2, 3, and 4 respectively.

Multiples

Multiples are *products* created by multiplying a base number times a series of numbers.

Base × Number = Multiple

Example: 2 × 3 = 6, so 6 is a multiple of base 2.



Imagine
multiples
as mounds
built from
a base.

Common Multiples are multiples that are the same for different bases.

Number Series

×	2	3	4	5	6
Base 2	4	6	8	10	12
Base 3	6	9	12	15	18

LCM

Multiples

6 and 12 are common multiples of the bases 2 and 3.
6 is the Least Common Multiple (LCM) of 2 and 3.
LCM is also known as the *Lowest* Common Multiple.
In fractions, the LCM is the LCD: Least Common Denominator.

Why Make Multiples?
One reason is to find a common number that several bases will dissolve into; e.g., a common denominator.

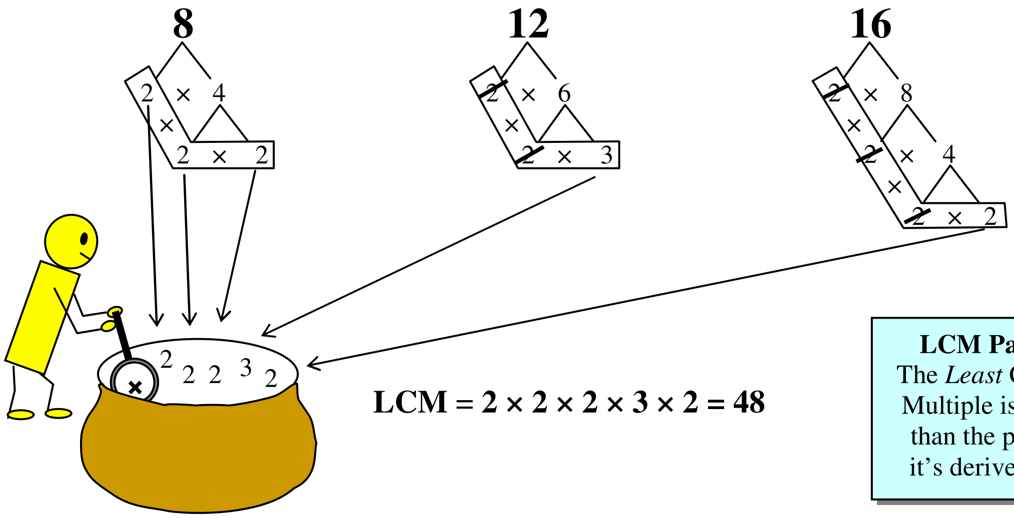
LCM: Load, Crush, Mix

To find the LCM of several products, factor each product into prime factors (p.10).

- Load *all* of the first product's prime factors into a large cooking pot.
- Crush (cross out) factors from the next product/s that are already in the pot. Load what's left.
- Mix (magnify/multiply) the factors in the cooking pot to get the LCM.

Example: Find the LCM of 8, 12, and 16.

The goal is to not include any more factors than are needed to dissolve each product into the cooking pot. Crushing eliminates redundant factors.



LCM = 2 × 2 × 2 × 3 × 2 = 48

LCM Paradox
The *Least* Common Multiple is *greater* than the products it's derived from.

Dissolving 8, 12, and 16 into the LCM of 48 results in 6, 4, and 3.
With denominators, the LCM is the LCD: Least Common Denominator.
Example of use: 1/8 + 1/12 + 1/16 = 6/48 + 4/48 + 3/48 = 13/48.

Algebra Basics

Term → Expression → Equation

Let’s compare what you already know about English parts of speech to Algebra terminology.

ENGLISH		ALGEBRA	
Word	John	**Term**	1
conjunction	and	+ or – operator	+
Phrase	John and Mary	**Expression**	1 + 1
verb	are	relational operator	=
Sentence	John and Mary are together.	**Equation**	1 + 1 = 2

TERM

A term is a mathematical *word*.

The + or – operators are mathematical *conjunctions* that join terms.

EXPRESSION

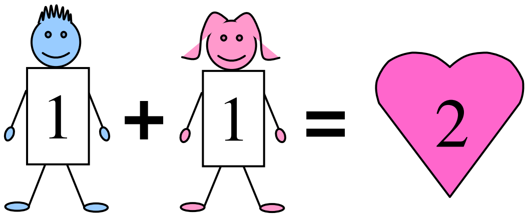
An expression is a mathematical *phrase* built from a term or terms.

The relational operators are mathematical *verbs* that join expressions.

EQUATION

An equation is a mathematical *sentence* that equates two expressions; e.g., 1 + 1 = 2

An *inequality* is a mathematical sentence that relates unequal expressions; e.g., 1 + 1 > 1



SENTENCE

	Phrase			Phrase
Word	Conjunction	Word	Verb	Word
John	**and**	**Mary**	**are**	**together.**

EQUATION

	Expression			Expression
Term	Operator	Term	Operator	Term
1	**+**	**1**	**=**	**2**

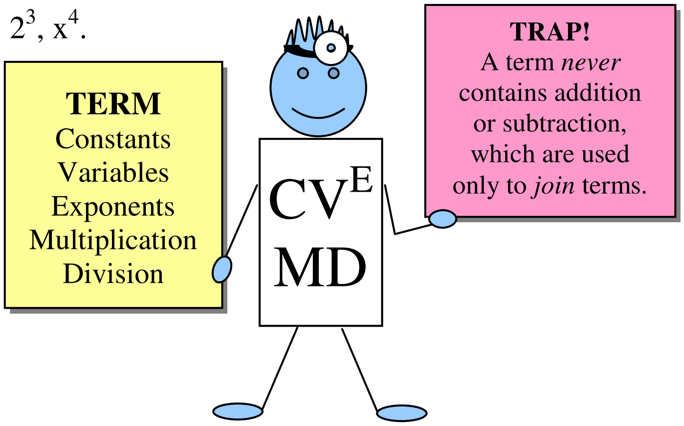
Term: CV^EMD

A term is a mathematical *word*.

English has different types of words: nouns, pronouns, etc. Similarly, math has different types of terms. A term can include any or all of the following components:

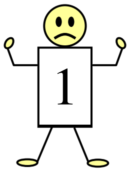
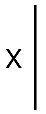
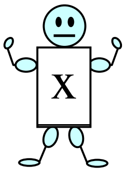
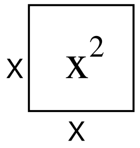
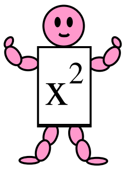
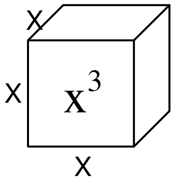
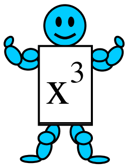
- **Constants**
  Numbers that do not vary; e.g., 100 is the number of cents in a dollar.
- **Variables**
  Letters that represent numbers that can vary; e.g., N is the number of cents in your penny jar.
- **Exponents**
  Powers assigned to constants or variables; e.g., 2^3, x^4.
- **Multiplication**
  Multiplied components; e.g., $3x^2y$.
- **Division**
  Divided components; e.g., $y^3/5$.

BrainAid
Acronym: CV^EMD
Acrostic: CardioVascular Expert—Medical Doctor



Term Families

Imagine that a term is like a person. As each person is a unique blend of body parts, each term is a unique blend of math components. A person belongs to a family whose power is determined by its wealth and social standing. A term belongs to a family whose power is determined by its exponent.

Power	Family	Visual	BrainAid
x^0 (equals 1)	Constant Term Family	1	I'm not very strong. 
x^1 (equals x)	Line Term Family		I'm strong. 
x^2	Square Term Family		I'm very strong. 
x^3	Cube Term Family		I'm extremely strong. 

Term Operators

Since 'x' is often used as a variable, avoid using it to show multiplication in algebra.

Instead use a dot, place items next to each other, or use parentheses:

$a \bullet b$

ab

$(a)(b)$

$a(b+c)$

Use fraction lines for division:

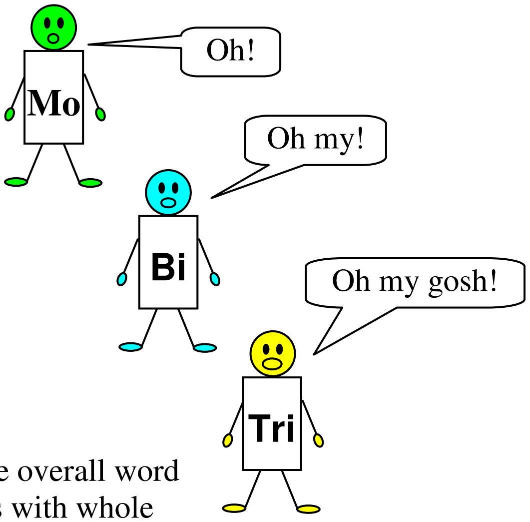
$\frac{a}{b}$ or a/b

Expression: Mono or Poly

An expression is a mathematical *phrase* built from a term or terms.

Expressions are classified by how many terms they contain.

EXPRESSION	Example
Monomial (moh-NOH-mee-ul)	$\mathbf{x^2}$
Binomial (bii-NOH-mee-ul)	$\mathbf{x^2 + x}$
Trinomial (trii-NOH-mee-ul)	$\mathbf{x^2 + x + 1}$



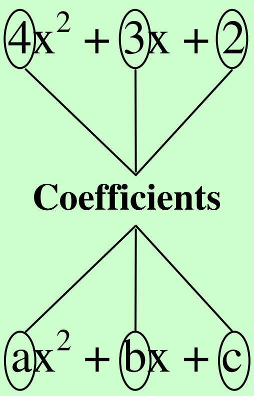
- Nomial means *name*, or in this case: *term.*
- Mono means *one.* A monomial has one term.
- Bi means *two*. A binomial has two terms.
- Tri means *three.* A trinomial has three terms.
- Poly means *many.* Polynomial (paw-lee-NOH-mee-ul) is the overall word for monomials, binomials, trinomials, and other expressions with whole number exponents (i.e., not negative like x^{-2} or fractional like $x^{1/2}$).

Coefficient Coworkers

Coefficients [coh-ee-FISH-untz] are constants coupled with variables.
Coefficients can be numbers, or letters that represent numbers.

Coefficient vs. Variable Letter Choices

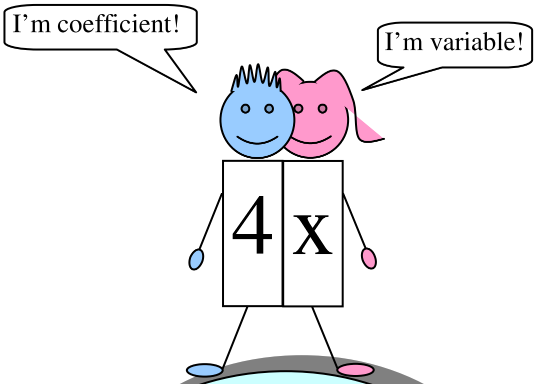
Coefficient letters are typically taken from the *beginning* of the alphabet (e.g., a, b, c). Although they are placeholders, coefficient letters represent constants, *not* variables. To avoid confusion, variable letters are typically taken from the *end* of the alphabet (e.g., x, y, z).



Note
The 2nd and 3rd terms contain x^1 and x^0 as in

$4x^2 + 3x^{\mathbf{1}} + 2\mathbf{x^0}$

But $x^1 = x$, so the exponent is omitted, and $x^0 = 1$, so there's no need to show it.

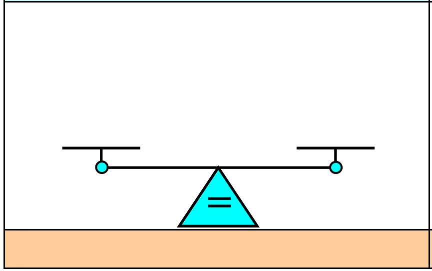
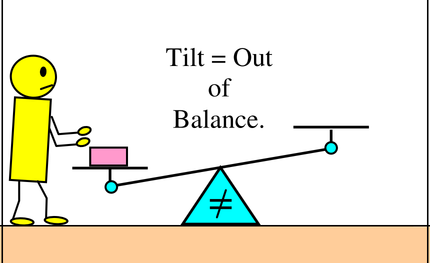
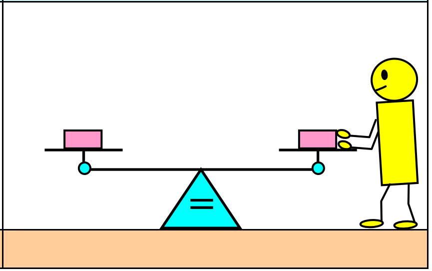


BrainAid
Coefficients are <u>co</u>workers that make variables more <u>efficient</u>.

Equation: Balancing Act

An equation is a mathematical *sentence* that equates two expressions.

An equation is like a balance scale that must have equal weight (expressions) on both sides to be balanced.

Start with an empty scale in a balanced condition (indicated by the = sign).	Add a weight to one side to unbalance the scale (indicated by the ≠ sign).	Add an equal weight to the other side to rebalance the scale (indicated by the = sign).
		

Golden Rule of Equations

Whatever you do to one side, do to the other side.

PROPERTY OF EQUALITY

If a = b, then

$$\mathbf{a + c = b + c}$$

You can add the same amount to both sides.

$$\mathbf{a - c = b - c}$$

You can subtract the same amount from both sides.

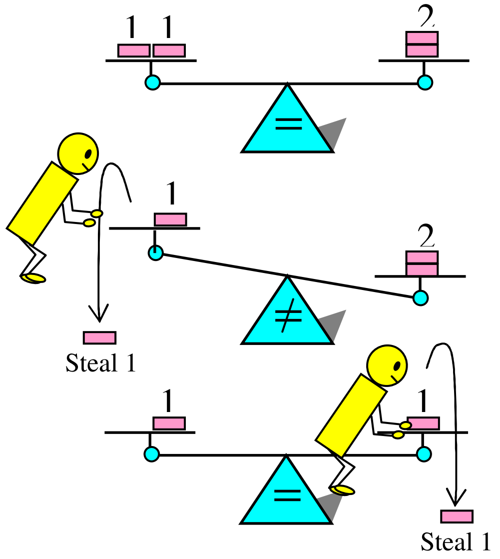
$$\mathbf{ac = bc}$$

You can multiply both sides by the same amount.

$$\mathbf{a/c = b/c}$$

You can divide both sides by the same amount.

To keep the scale or equation balanced, whatever you do to one side, you must do to the other side.



$$\begin{array}{l} 1+1=2 \\ \underline{-1} \\ 0+1\neq 2 \\ \qquad\quad \underline{-1} \\ 1 \qquad = 1 \end{array}$$

Algebra: Science of Equations

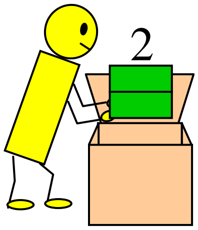
Algebra is the branch of mathematics that uses equations to join expressions.
Algebra [AL-jeh-bruh] comes from *Al Jabr,* which is Arabic for “bringing together.”

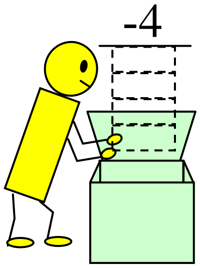
Most people are comfortable with arithmetic. So why do they panic when it comes to algebra? In a word: variables. *How strange that letters, which seem so natural and non-threatening when used for words, become frightening when used with numbers. To be successful with algebra, you must make friends with variables.*

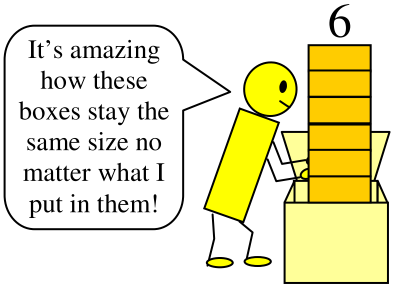
Variable = Box

A variable is a letter used as a placeholder for a number that can vary.
Variables are sometimes referred to as *unknowns*, since they represent unknown numbers.
Variables are also known as *literal* numbers. In this case, literal means “letter.”

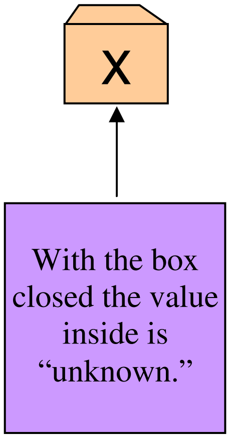
If we knew all the numbers in a problem beforehand, there would be no need for variables. But in real-life situations, there’s usually something we’re trying to discover. Variable placeholders allow us to manipulate an equation until we discover the unknown numbers. Imagine that variables are magic boxes that can hold any number—positive, negative, small, or large. Like a genie fitting into a bottle, even a large number can be put into a box without changing its size.

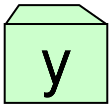


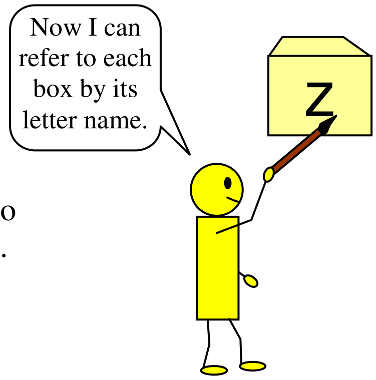




Imagine painting letters on variable boxes so we can identify them by name.
The most common letter used is **x**, but we can use any letter, upper or lowercase.







Variable letters are equal to whatever is in their boxes.

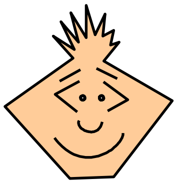
x = 2

y = -4

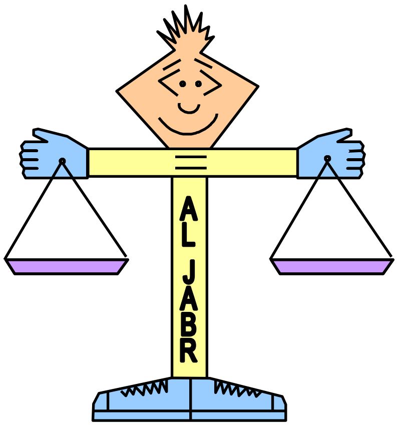
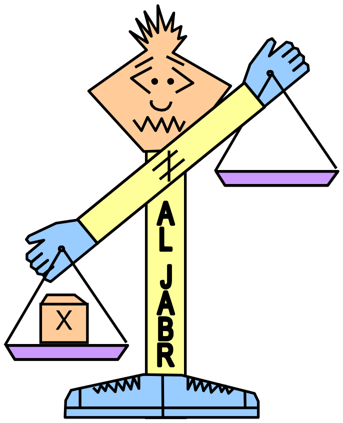
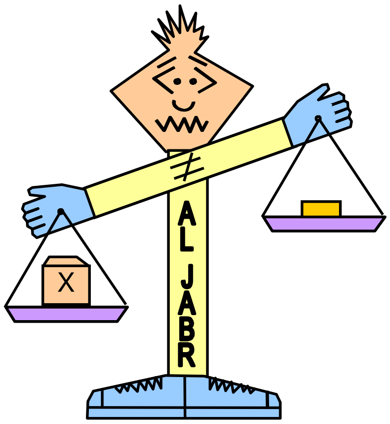
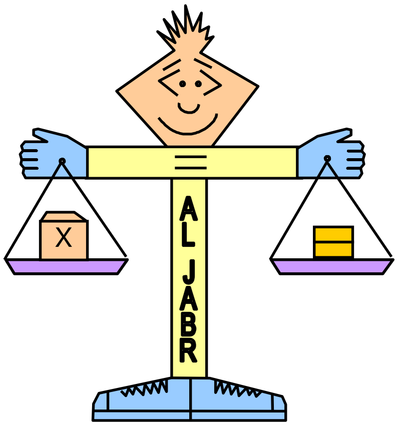
z = 6

Goal of Algebra: What's in the Box?

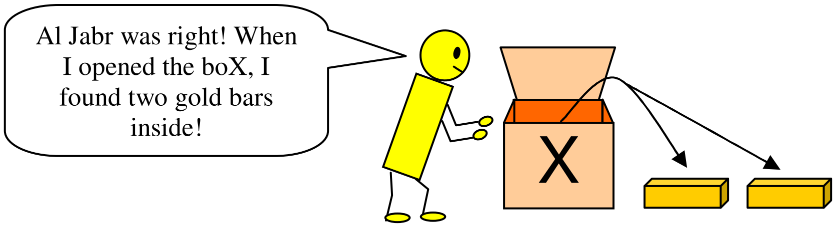
Al Jabr says: *What's in the box?*



Hi, I'm Al Jabr. My name is Arabic for "bringing together," and that's what algebra does. It brings together something we don't know (unknown) with something we do know (given). Algebra uses the given value to find the unknown value.

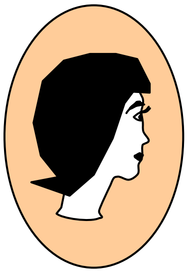
My body is shaped like a scale. When my arms are balanced, I have a big smile on my face!	I want to find out how many gold bars are in this treasure boX. It's rusted shut and sure is heavy!
	
One gold bar helps, but isn't quite enough to balance me out. Let's try one more.	That's much better. I'm balanced now, so the treasure boX must contain two gold bars!
	

Al Jabr cleverly adjusted his arms to counter the weight of the boX, so the only thing he was measuring for was the weight of what was *inside* the boX. Using algebraic [al-jeh-BRAA-ik] language, we can state the problem this way: How many gold bars (*unknown*) are in the treasure boX, *given* that its contents are balanced by two gold bars? The obvious answer is two gold bars.



Isolating the Variable: Garbo Rule

Greta Garbo says: *I vant to be alone!*



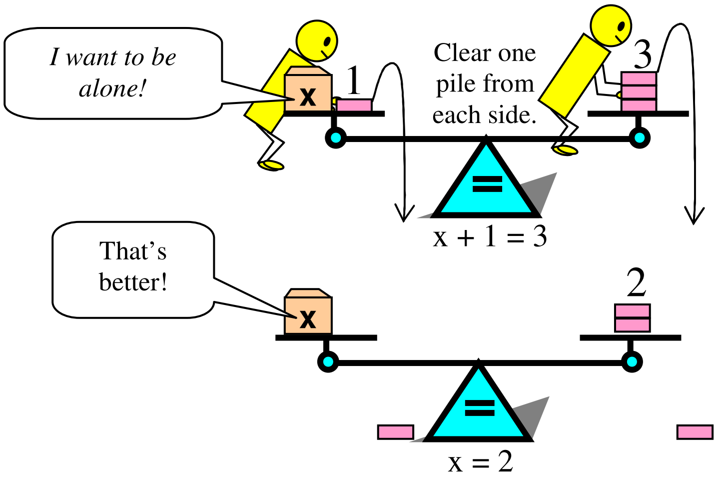
Greta Garbo was a movie star in the 1930s who came to shun publicity. She once complained in her foreign accent: *I vant to be alone!*

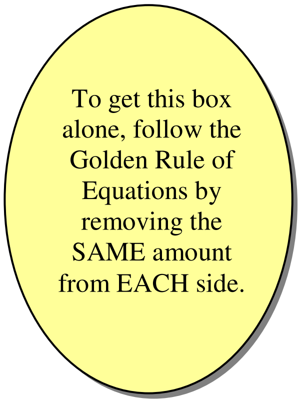
We'll use Ms. Garbo's famous lament as a BrainAid, because the variable box also "wants to be alone." Our goal is to get the box alone on one side of the scale, so that it's contents are revealed on the opposite side.

Isolating the Variable: Clearly Opposite

To isolate the variable, *clear* everything away from it with an *opposite* (aka reciprocal) operation.

- If a term is *added* to the variable side, clear it by *subtracting* it from both sides.
- If a term is *subtracted* from the variable side, clear it by *adding* it to both sides.
- If the variable is *multiplied* by a coefficient, clear it by *dividing* both sides by the coefficient.
- If the variable is *divided* by a coefficient, clear it by *multiplying* both sides by the coefficient.



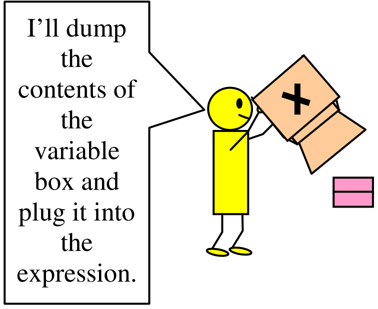


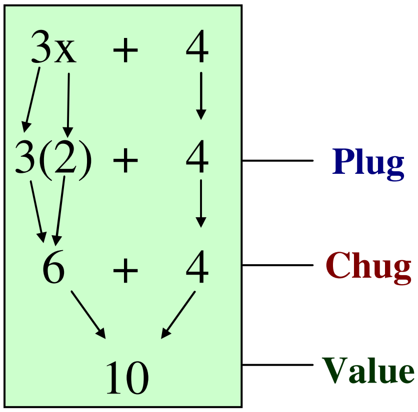
Evaluating an Expression: Plug & Chug

To evaluate means to "find the value of" an expression given the value/s of its variable/s.

1. *Plug* in (substitute) the given value/s for the variable/s.
2. *Chug* ahead and perform the operation.

Example: If x = 2, what is the value of 3x + 4?



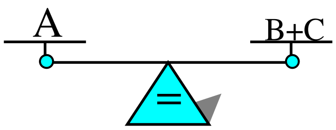
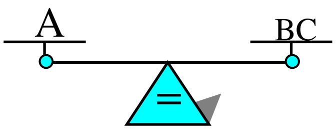
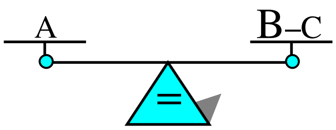
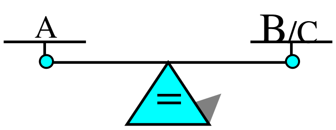
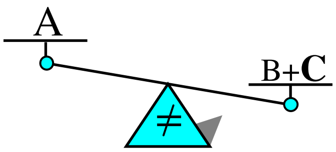
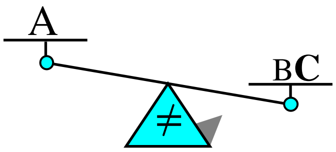
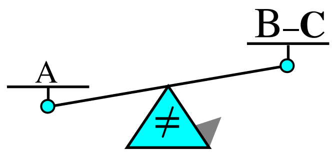
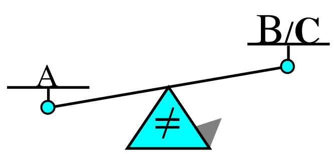
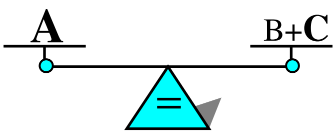
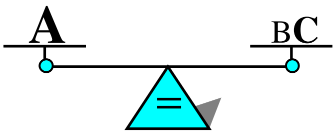
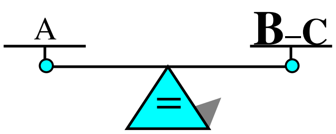
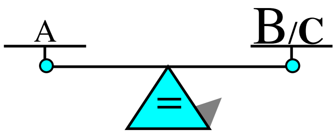


Proportionality

In equations, proportionality affects how changing one item affects another.

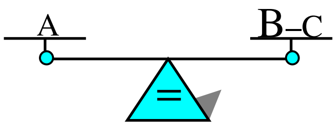
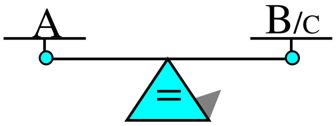
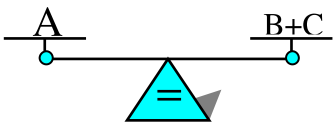
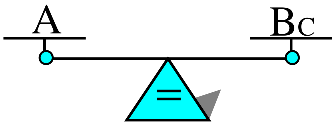
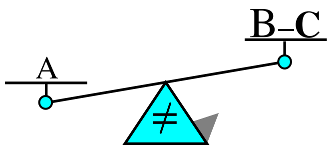
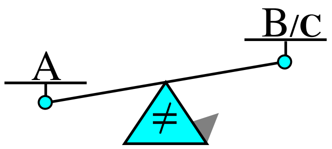
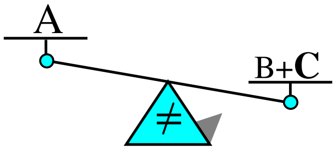
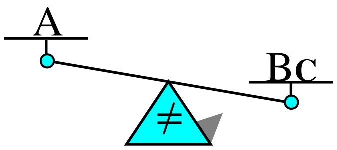
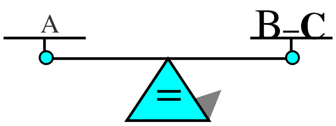
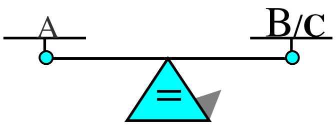
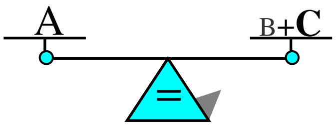
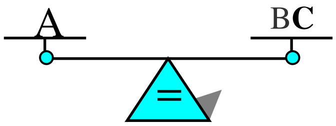
Proportional

Proportional [proh-POR-shun-ul] items increase (or decrease) *together* to keep the equation balanced.

Opposite Sides As **C** increases, **A** increases.		**Same Side** As **C** increases, **B** increases.	
Addition	Multiplication	Subtraction	Division
 $4 = 2 + 2$	 $4 = 2 \bullet 2$	 $2 = 4 - 2$	 $2 = 4 / 2$
 $4 \neq 2 + \mathbf{3}$	 $4 \neq 2 \bullet \mathbf{3}$	 $2 \neq 4 - \mathbf{3}$	 $2 \neq 4 / \mathbf{3}$
 $\mathbf{5} = 2 + \mathbf{3}$	 $\mathbf{6} = 2 \bullet \mathbf{3}$	 $2 = \mathbf{5} - \mathbf{3}$	 $2 = \mathbf{6} / \mathbf{3}$

Inversely Proportional

Inversely Proportional items increase (or decrease) *oppositely* to keep the equation balanced.

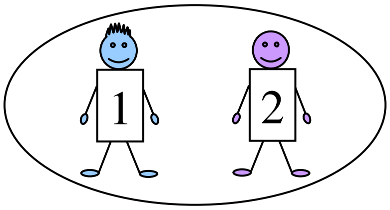
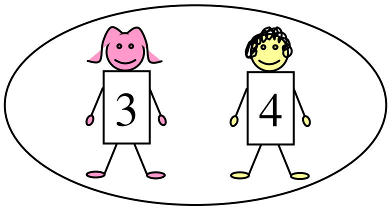
Opposite Sides As **C** increases, **A** decreases.		**Same Side** As **C** increases, **B** decreases.	
Subtraction	Division	Addition	Multiplication
 $2 = 4 - 2$	 $4 = 4 / 1$	 $4 = 2 + 2$	 $4 = 4 \bullet 1$
 $2 \neq 4 - \mathbf{3}$	 $4 \neq 4 / \mathbf{2}$	 $4 \neq 2 + \mathbf{3}$	 $4 \neq 4 \bullet \mathbf{2}$
 $1 = 4 - \mathbf{3}$	 $2 = 4 / \mathbf{2}$	 $4 = 1 + \mathbf{3}$	 $4 = 2 \bullet \mathbf{2}$

Relation: Pairing Up

A relation is a collection of ordered pairs.

Ordered Pair: (Boy, Girl)

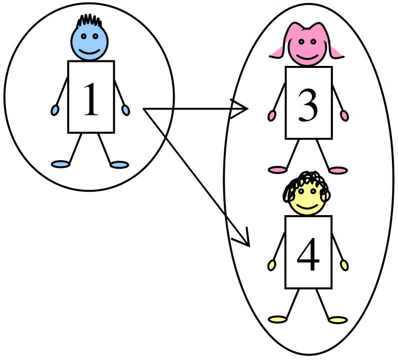
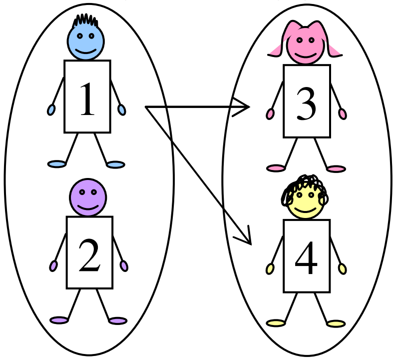
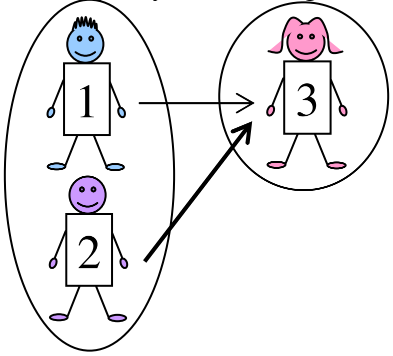
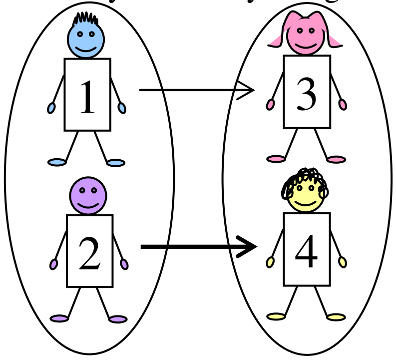
An Ordered Pair is made of two numbers written inside parentheses in this order: (Domain, Range).

DOMAIN	RANGE
The first numbers of a collection of ordered pairs make up the Domain of the relation.  **Domain Numbers = Independent Boys** Boys enter ordered pairs first, alone and independent (1,) (2,).	The second numbers of a collection of ordered pairs make up the Range of the relation.  **Range Numbers = Dependent Girls** Girls enter ordered pairs second, dependent upon which boy is there (1,3) (2,4).

BrainAid: (Domain, Range) is in alphabetical order; i.e., D comes before R.
It is also in the same order as the musical scale (Do, Re).

Types of Relations: Dating

Imagine boys and girls forming relations for dating.

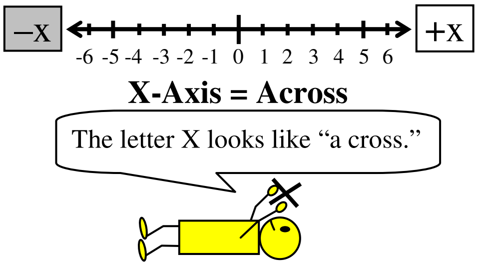
One-to-Many: (1,3)(1,4) One boy dates many girls.  	**Many-to-Many: (1,3)(1,4)(2,3)(2,4)** Each boy dates many girls. 
Many-to-One: (1,3)(2,3) Many boys date one girl.  	**One-to-One: (1,3)(2,4)** Each boy dates only one girl. 

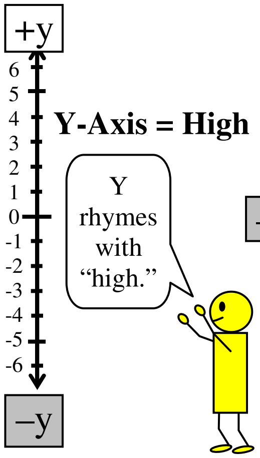
Cartesian Coordinates: (x, y)

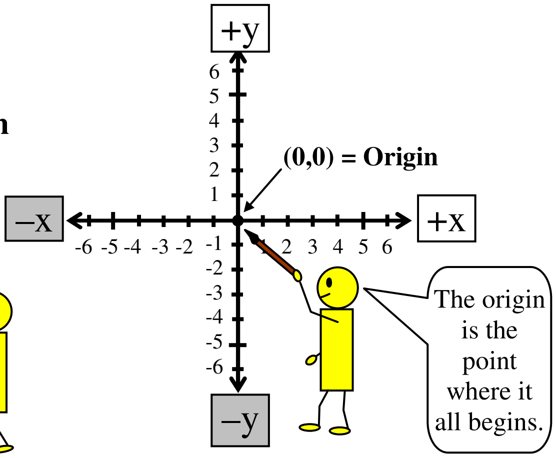
Cartesian coordinates [car-TEE-zhun koh-OR-di-nutz] are ordered pairs represented by (x, y) displayed on a two-dimensional graph. The word 'Cartesian' comes from Rene Descartes [reh-NAA daa-KART], the 17th century French philosopher/mathematician who conceived the system.

Axes

Two number lines, called *axes* [AX-eez], intersect at a point called the *origin* [OR-ih-jin], which corresponds to the ordered pair (0,0).







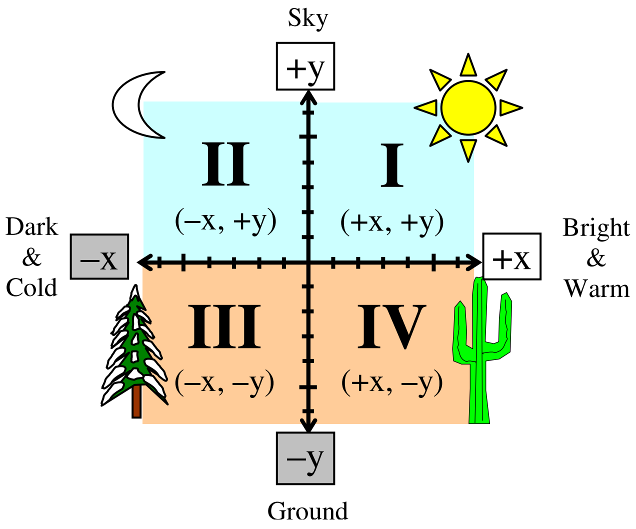
Quadrants

The x-axis [AX-iss] and the y-axis create four quadrants [KWAW-druntz] or quarters.

BrainAid

Imagine the x-axis divides ground from sky.
Imagine the y-axis divides night from day.
Positive is bright and warm.
Negative is dark and cold.

I. Upper right: Day sky (+x, +y)
II. Upper left: Night sky (–x, +y)
III. Lower left: Cold ground (–x, –y)
IV. Lower right: Warm ground (+x, –y)



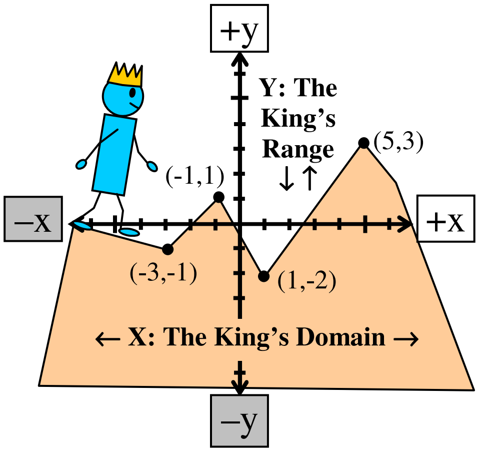
Coordinates

X-Coordinate: The variable **x** independently goes right or left *across* the Domain (think region or territory).

Y-Coordinate: The variable **y,** whose value is dependent upon **x**, goes *high* up/down the Range (think mountain).

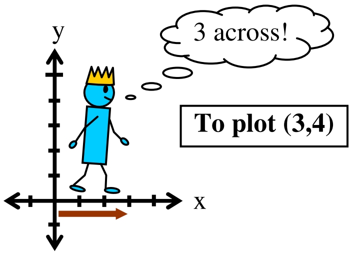
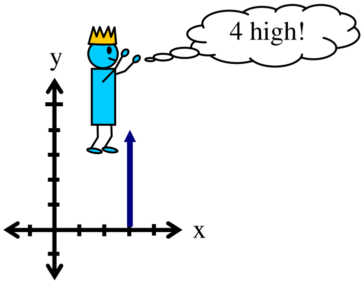
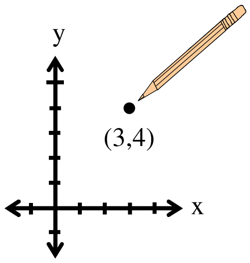
BrainAid: Imagine a king hiking *independently* across his *domain*. His elevation in the mountain *range* is *dependent* on his position in his domain.

The King's travels: At (-3,-1) he is 3 left and 1 down from the (0,0) origin, which is the center of his kingdom. At (-1,1) he is 1 left and 1 up. At (1,-2) he is 1 right and 2 down. At (5,3) he is 5 right and 3 up from the origin.



Plotting Ordered Pairs: x across, y high

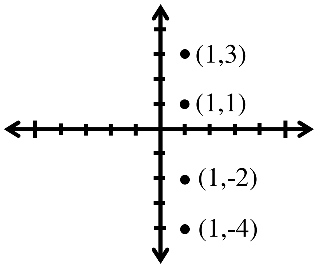
To "plot" an ordered pair means to find then draw and label a point on a Cartesian-Coordinate graph.

To plot (x,y) stand at the origin (0,0) then:		
Think x is across! If **x** is positive, step right. If **x** is negative, step left.	**Think y is high!** If **y** is positive, leap up. If **y** is negative, dig down.	**Make A Point!** Draw a dot and label it with the ordered pair.
		

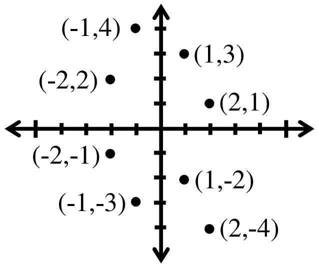
Plotting Relations: All over the map

Compare each plot below to the type of relation shown on page 20.

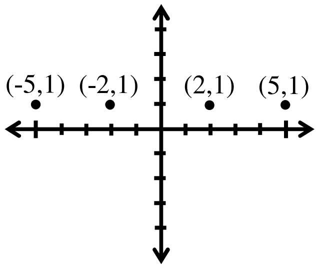
One x to Many y



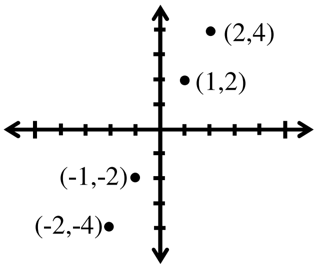
Many x to Many y



Many x to One y

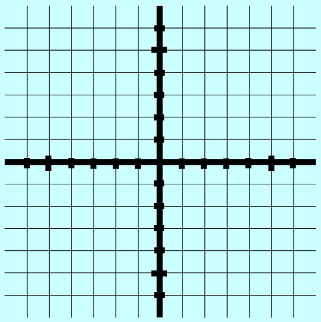


One x to One y



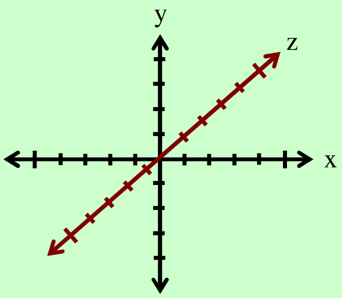
Tip

When plotting coordinates, it's easier and more accurate to use graph paper with preprinted grids.



3-D!

FYI: With the addition of a Z-axis running front-to-back, coordinates can be plotted in *three* dimensions. (x, y, z).

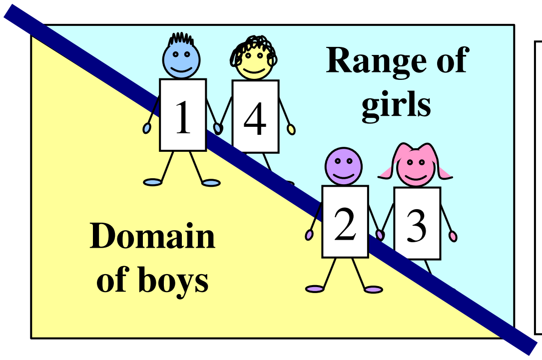


Function: Fun at the Junction

A function is a *relation* where each domain value x has only *one* range value y.

The value of y is *a function of* (i.e., depends upon) the value of x.

BrainAid
Sets of boys (domain) and girls (range) have fun at the Junction where they meet for dates.



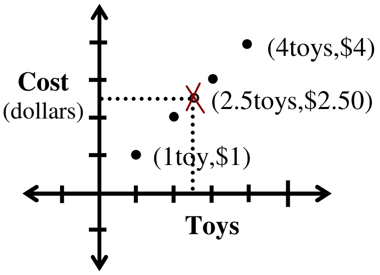
BrainAid
The place where a dependent girl meets an independent boy is *a function of* where he stands along the Junction.

Discrete vs. Continuous: No line vs. line

Discrete Function plots consist of separate points that *cannot* be connected by a line.

Discrete means separate or distinct. Splitting discrete items (e.g., people, objects, places) into smaller pieces doesn't make sense.

Example: The total cost (range) of toys that cost $1 each is a function of the number of toys (domain) purchased.

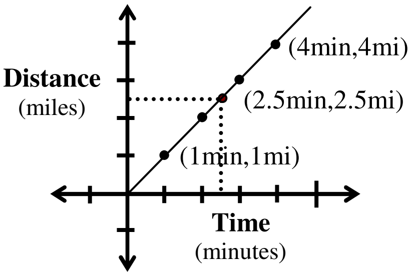


Although it looks like a straight-line relationship, buying 2.5 toys doesn't make sense.

Continuous Function plots consist of an unbroken line of points.

Continuous items (e.g., time, speed, distance, temperature) can be reasonably split into smaller pieces.

Example: The distance (range) a vehicle going 1 mile/minute travels is a function of the time (domain) it has traveled.



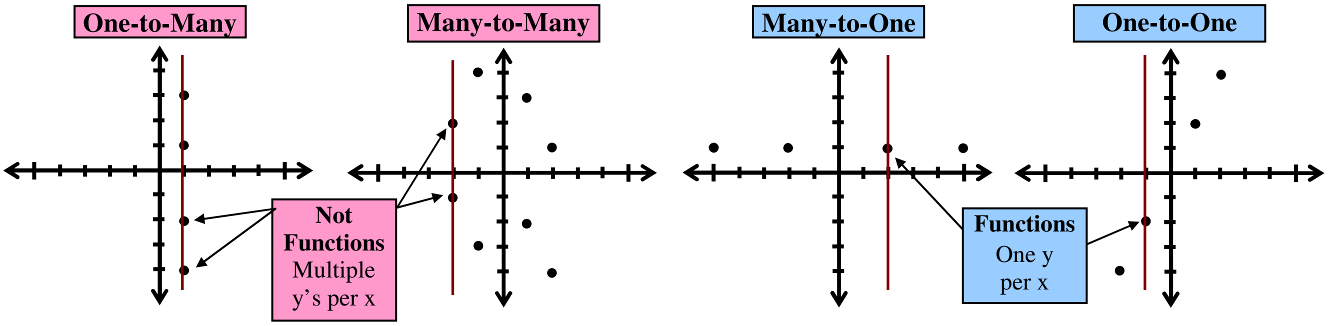
Any point along the line satisfies the function.

Traveling for 2.5 minutes makes sense.

Vertical Line Test: One y per x

If a vertical line can be drawn through two or more points of a relation, it's *not* a function.

Explanation: Since, by definition, each domain value in a function can have only ***one*** range value, functions are limited to Many-to-One or One-to-One relations (p.20).



Standard Function Layout: y = x

Range variable = Domain expression

y = x expression

Isolating the **y variable** on the left makes it easy to see that it's *a function of* (i.e., depends upon) the **x expression** on the right.

If x changes, y changes.

Example: $y = x + 1$
If $x = 2$, then $y = (2) + 1 = 3$
If $x = 3$, then $y = (3) + 1 = 4$

- or -

Reverse-Order Paradox
Ordered Pair
(x, y)
y = x
Function

f(x) = x expression

Using **f(x)** in place of **y** has the advantage of showing the x value that produces the range value, e.g.,

$$f(x) = x + 1$$
$$f(2) = 2 + 1$$
$$f(2) = 3$$

f(x), pronounced *f of x*, means "function of x," *not* "f times x."

Function Families

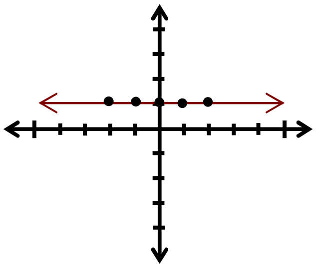
Functions, like the terms they contain, can be classified into families based on the power of their exponents (see Term Families p.13). Each function family has a different shape when graphed. The arrows on the ends of lines and curves indicate that they continue forever—to infinity!

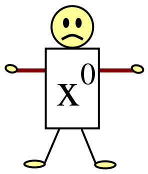
Constant [KAWN-stunt] Function Family

Flat Line

$y = x^0$

x	y
-2	1
-1	1
0	1
1	1
2	1



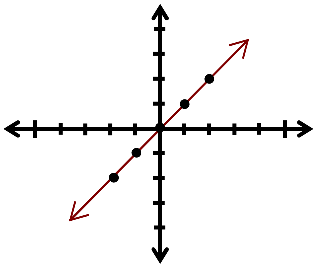


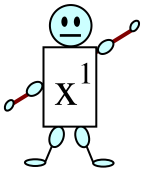
Linear [LIH-nee-ur] Function Family

Sloped Line

$y = x^1$

x	y
-2	-2
-1	-1
0	0
1	1
2	1



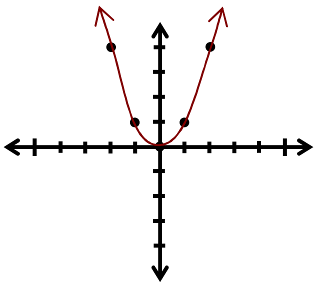


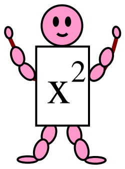
Quadratic [kwaw-DRA-tik] Function Family

Parabola [puh-RA-boh-luh]

$y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4





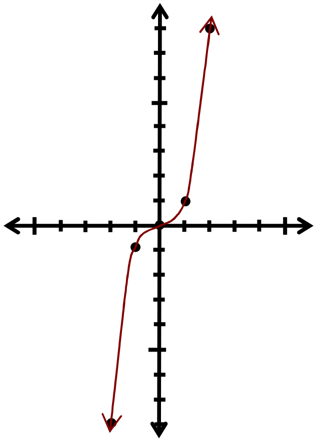
BrainAid
A parabola is bowl-shaped.

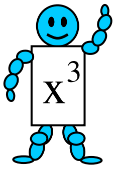
Cubic [KYU-bik] Function Family

Curve

$y = x^3$

x	y
-2	-8
-1	-1
0	0
1	1
2	8





BrainAid
A cubic curve waves "hi!"

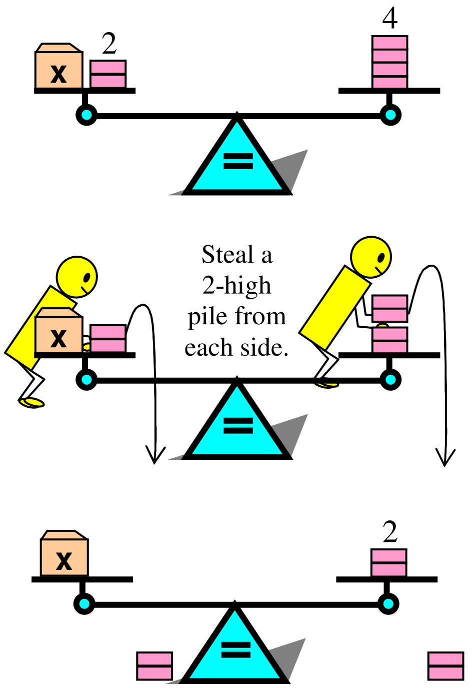
Operations

One Equation, One Unknown

The simplest algebra problems have One Equation with One first-power (x^1) Unknown.
For short, we'll call these 1EqUnk [ek-unk] problems.

1EqUnk Added Term (x + 2 = 4)

Goal: What's in the box?
Garbo Rule: Get the box alone.
Clearly Opposite: Subtract the added amount from each side.



Solve

$$x + 2 = 4$$

Subtract 2
from each side

$$x + \cancel{2} = 4$$
$$\underline{-\cancel{2}} \quad \underline{-2}$$

These
cancel

$$x \qquad = 2$$

Important!
Plug the solution back into
the original equation to
make sure it's correct.

Check

x + 2 = 4
2 + 2 = 4
4 = 4 √

Tips
To avoid errors,
keep terms and
operators lined up.

Circle the
solution.

Your turn: Solve for x by subtracting.

Solve
x + 3 = 4

Check
x + 3 = 4

Solve
x + 2 = 5

Check
x + 2 = 5

Solve
x + 5 = 9

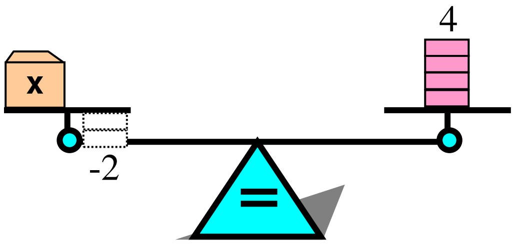
Check
x + 5 = 9

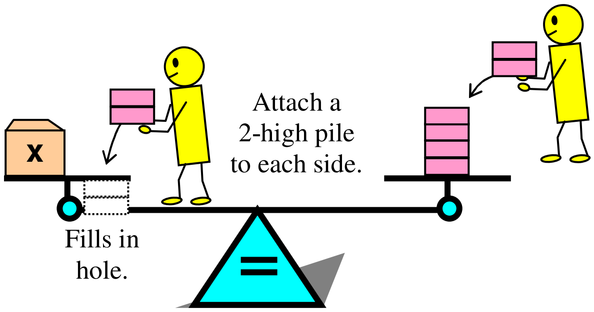
Solve
x + 6 = 15

Check
x + 6 = 15

1EqUnk Subtracted Term (x – 2 = 4)

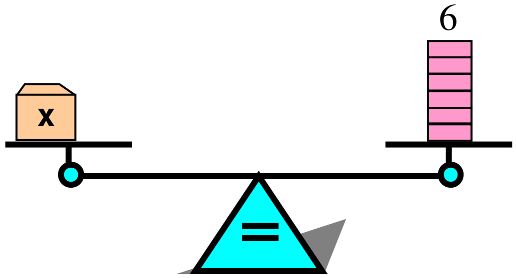
Goal: What's in the box?
Garbo Rule: Get the box alone.
Clearly Opposite: Add the subtracted amount to both sides.





Attach a 2-high pile to each side.

Fills in hole.



Solve

$x - 2 = 4$

Add 2 to each side

$$x - \cancel{2} = 4$$
$$\underline{+\cancel{2}} \quad \underline{+2}$$

These cancel

$$x \quad = 6$$

Important!
Plug the solution back into the original equation to make sure it's correct.

Check

$x - 2 = 4$
$\mathbf{6} - 2 = 4$
$4 = 4$ √

Tips
To avoid errors, keep terms and operators lined up.

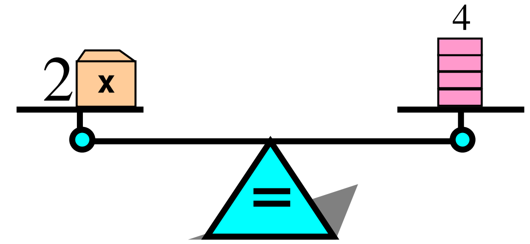
Circle the solution.

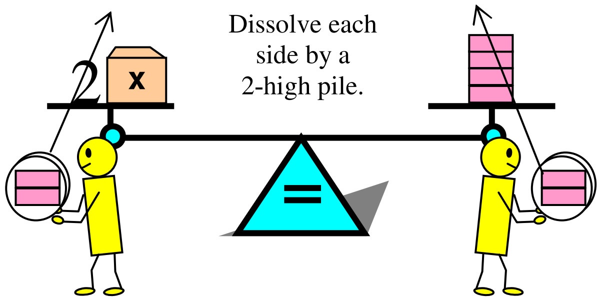
Your turn: Solve for x by adding.

Solve	**Check**	**Solve**	**Check**
$x - 3 = 4$	$x - 3 = 4$	$x - 4 = 5$	$x - 4 = 5$
Solve	**Check**	**Solve**	**Check**
$x - 5 = 9$	$x - 5 = 9$	$x - 6 = 15$	$x - 6 = 15$

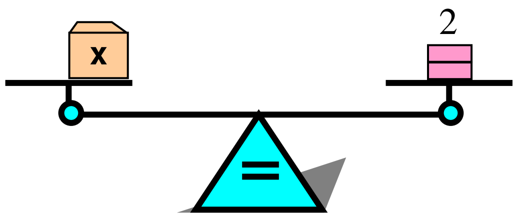
1EqUnk Multiplied Variable (2x = 4)

Goal: What's in the box?
Garbo Rule: Get the box alone.
Clearly Opposite: Divide each side by the multiplied amount.





Dissolve each side by a 2-high pile.



Solve

$$2x = 4$$

Divide each side by 2

$$\frac{\overset{1}{\cancel{2}}x}{\underset{1}{\cancel{2}}} = \frac{4}{2}$$

These dissolve

$$\boxed{x = 2}$$

Important!
Plug the solution back into the original equation to make sure it's correct.

Check

2**x** = 4
2(**2**) = 4
4 = 4 √

Tips
To avoid errors, keep terms and operators lined up.

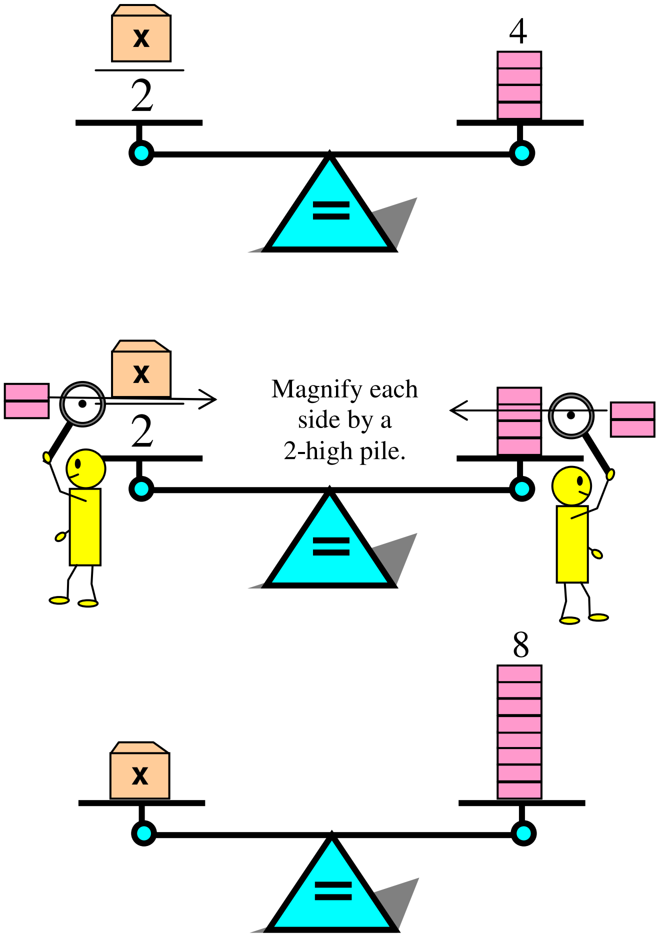
Circle the solution.

Your turn: Solve for x by dividing.

Solve	**Check**	**Solve**	**Check**
2x = 6	2x = 6	3x = 6	3x = 6
4x = 20	4x = 20	5x = 20	5x = 20

1EqUnk Divided Variable (x/2 = 4)

Goal: What's in the box?
Garbo Rule: Get the box alone.
Clearly Opposite: Multiply each side by the divided amount.



Solve

$$\frac{x}{2} = 4$$

Multiply each side by 2.

$$\frac{(\cancel{2}^{1})x}{\cancel{2}_{1}} = 4(2)$$

These dissolve

$$\boxed{x = 8}$$

Important!
Plug the solution back into the original equation to make sure it's correct.

Check

$\frac{x}{2} = 4$

$\frac{8}{2} = 4$

$4 = 4$ √

Tips
To avoid errors, keep terms and operators lined up.

Circle the solution.

Your turn: Solve for x by multiplying.

Solve
$\frac{x}{2} = 6$

Check
$\frac{x}{2} = 6$

Solve
$\frac{x}{3} = 1$

Check
$\frac{x}{3} = 1$

Solve
$\frac{x}{4} = 2$

Check
$\frac{x}{4} = 2$

Solve
$\frac{x}{5} = 3$

Check
$\frac{x}{5} = 3$

Multiple Operations: Clear As Mud

When an equation contains multiple operators, it may not be clear what you should do first. In fact, it's as *clear as mud!*

1st Clear away any term/s Added or Subtracted to the variable.
2nd Clear away any coefficient/s from a Multiplied or Divided variable.

BrainAid: It's as clear A/S M/D. Added/Subtracted; Multiplied/Divided.

Solve

$$3x + 7 = 13$$

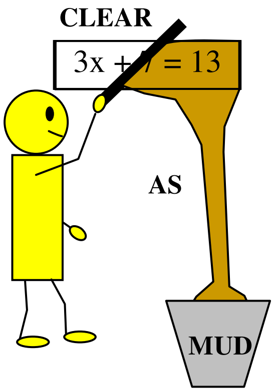
Clear Added term by subtracting 7 from both sides.

Clear Multiplied variable by dividing both sides by 3.

$$\begin{array}{rcl} 3x + \cancel{7} & = & 13 \\ \underline{-\cancel{7}} & & \underline{-7} \\ \dfrac{\cancel{3}x}{\cancel{3}} & = & \dfrac{6}{3} \\ \boxed{x} & \boxed{=} & \boxed{2} \end{array}$$

Check

$$\begin{array}{rcl} 3(\mathbf{2}) + 7 & = & 13 \\ 6 + 7 & = & 13 \\ 13 & = & 13 \ \surd \end{array}$$



Exception to A/S M/D Order
With some problems, you'll want to first "throw mud" (multiply or divide) to clear away or reduce coefficients before clearing added or subtracted terms (see p.32).

Solve

$$\frac{x}{2} - 5 = 1$$

Clear Subtracted term by adding 5 to both sides.

Clear Divided variable by multiplying both sides by 2.

$$\begin{array}{rcl} \dfrac{x}{2} - \cancel{5} & = & 1 \\ \underline{+\cancel{5}} & & \underline{+5} \\ (\cancel{2})\dfrac{x}{\cancel{2}} & = & 6(2) \\ \boxed{x} & \boxed{=} & \boxed{12} \end{array}$$

Check

$$\begin{array}{rcl} \dfrac{\mathbf{12}}{2} - 5 & = & 1 \\ 6 - 5 & = & 1 \\ 1 & = & 1 \ \surd \end{array}$$

Your turn: Solve using the Clear-As-Mud procedure.

Solve

$$2x - 3 = 7$$

Check

Solve

$$\frac{x}{3} + 1 = 2$$

Check

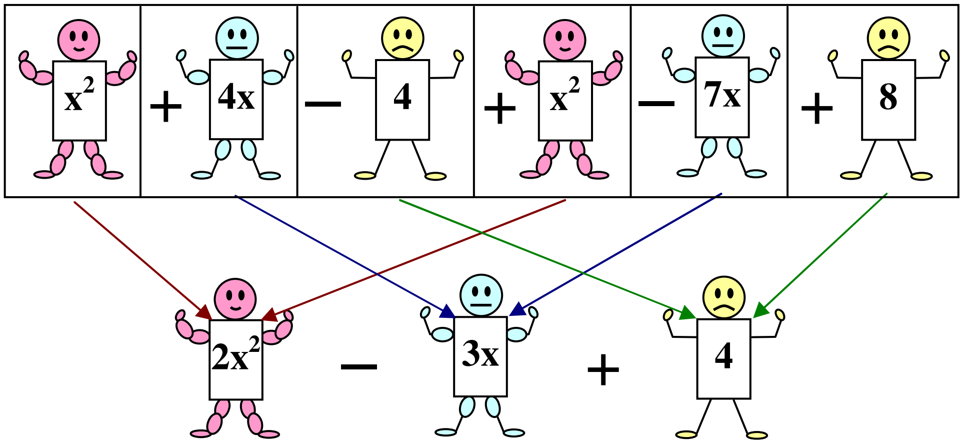
Simplifying Terms

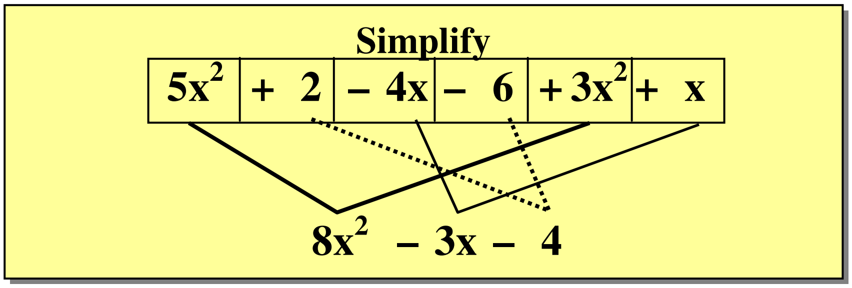
Multiple Terms: Family Reunion

In expressions with multiple terms, combine *like* terms.
Like (aka similar) terms have the same variable/s raised to the same power/s (Term Families p.13).

Draw a box around each term, *including its sign*.
Draw lines from like terms, and combine them into single terms.
Place the highest power term on the left and proceed in descending order: x^2, x^1, x^0.

BrainAid: Imagine like terms combining together at family reunions.
Each family's value is a mix of the positive and negative personalities (coefficients) of its members.





Simplify

$$\boxed{5x^2}\boxed{+\ 2}\boxed{-\ 4x}\boxed{-\ 6}\boxed{+3x^2}\boxed{+\ x}$$

$$8x^2\ -3x\ -\ 4$$

Your turn: Simplify the expressions by holding Family Reunions.

Simplify $4x - 6\ + 3x +\ 1$	**Simplify** $-4x +\ 6\ + 2x +\ 3\ +\ x^2$
Simplify $2x + 7\ \ - 4x^2 -\ 6\ + 3x +\ x^2$	**Simplify** $5x + 2\ \ - 7x^2 - 6x + 3x^2 +\ 1$

Separated Terms: Take Sides

If like terms are on opposite sides of the equal sign, move them to the same side and combine.

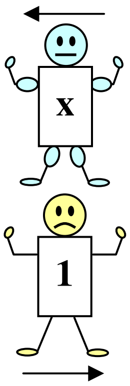
Simplify

$$3x + 2 = 4 - 2x$$

Move variable left.

Move constant right.

$$\begin{array}{rcl} 3x + 2 &=& 4 - \cancel{2x} \\ \underline{+2x} && \underline{+\cancel{2x}} \\ 5x + \cancel{2} &=& 4 \\ \underline{-\cancel{2}} && \underline{-2} \\ 5x &=& 2 \end{array}$$



Your turn: Simplify

$$4x - 3 = 5 + 2x$$

Which Side?

It's traditional to move variables to the left side, but not mandatory.
For example, **x = 1** is the same as **1 = x**

It's preferable to move variables to the side that results in a positive coefficient.

$$\begin{array}{rcl} \cancel{x} + 1 &=& 2x \\ \underline{-\cancel{x}} && \underline{-x} \\ 1 &=& x \end{array}$$

Distributed Terms: Fair to All

If terms in parentheses are multiplied by an outer term, distribute equally to every inner term.
Take special care to distribute negatives correctly.

Distribute to each inner term

$$4(x - 2) = 20$$
$$4x - 8 = 20$$

Distribute the negative number

$$-4(x - 2) = 20$$
$$-4x + 8 = 20$$

Distribute the minus sign

$$-(x - 2) = 20$$
$$-x + 2 = 20$$

This is equivalent to multiplying by –1.

Your turn:

Distribute to each inner term

$$5(-x + 2) = 25$$

Distribute the negative number

$$-5(-x + 2) = 25$$

Distribute the minus sign

$$-(-x + 2) = 25$$

Simplifying Coefficients: Throw Mud

To simplify coefficients, it may help to "throw a little mud" at them first.

Clear Denominators: Magnify by LCM

To clear constants or variables that appear in denominators, *throw mud* to multiply *all* terms by the LCM (p.11). This is the same as multiplying each side by the same amount, so the equation remains equal. If only one term has a denominator, it's the LCM, so multiply all terms by it.

Simplify

$$\frac{x}{2} + 3 = 4$$

Magnify by LCM of 2.

$$2\left[\frac{x}{2} + 3 = 4\right]$$

$$x + 6 = 8$$

Simplify

$$\frac{x}{2} + \frac{1}{8} = \frac{3}{4}$$

Magnify by LCM of 8.

$$8\left[\frac{x}{2} + \frac{1}{8} = \frac{3}{4}\right]$$

$$4x + 1 = 6$$

BrainAid
Throw M/D
means to
Multiply/Divide.

Your turn: Simplify

$$x + 1 = \frac{2}{5}$$

Your turn: Simplify

$$\frac{x}{6} + \frac{1}{2} = \frac{2}{3}$$

Reduce Coefficients: Dissolve with GCF

To reduce coefficients, *throw mud* by dividing the GCF (p.10) into each term.
If the variable coefficient is negative, divide by a negative GCF.

Simplify

$$4x + 8 = 16$$

Dissolve with GCF of 4.

$$\frac{4x + 8 = 16}{4}$$

$$x + 2 = 4$$

Simplify

$$-2x + 6 = -4$$

Dissolve with GCF of -2 to make the x coefficient positive.

$$\frac{-2x + 6 = -4}{-2}$$

$$x - 3 = 2$$

Your turn: Simplify

$$6x + 2 = 8$$

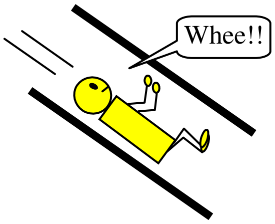
Your turn: Simplify

$$-3x - 6 = 9$$

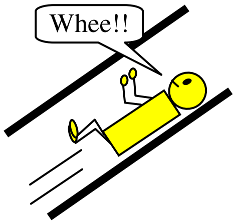
Fractional Terms

Clearing Equated Fractions: Shoot-the-Chute

When fractional expressions are set equal to each other,
cross multiply to clear their fractions and isolate the variable.



BrainAid: Shoot-the-Chute is an amusement park ride that has a chute or slide. Imagine that the equal sign between expressions is a chute that tilts so numbers and variables can "shoot" up or down through it.



Solve

$$\frac{3}{4} = \frac{2}{x}$$

Tilt chute. Shoot 3 down. Shoot x up.

$$\frac{3}{4} \searrow \frac{2}{x}$$

Tilt chute. Shoot 4 up.

4 • 2 = 8

$$\frac{x}{4} \nearrow \frac{2}{3}$$

Un-tilt chute.

$$x = \frac{8}{3}$$

Why It Works

Multiply both sides by x

$$(x)\frac{3}{4} = \frac{2(\cancel{x})}{\cancel{x}}$$

$$\frac{3x}{4} = 2$$

Divide both sides by 3

$$\frac{\cancel{3}x}{(\cancel{3})4} = \frac{2}{3}$$

$$\frac{x}{4} = \frac{2}{3}$$

Multiply both sides by 4

$$(\cancel{4})\frac{x}{\cancel{4}} = \frac{2(4)}{3}$$

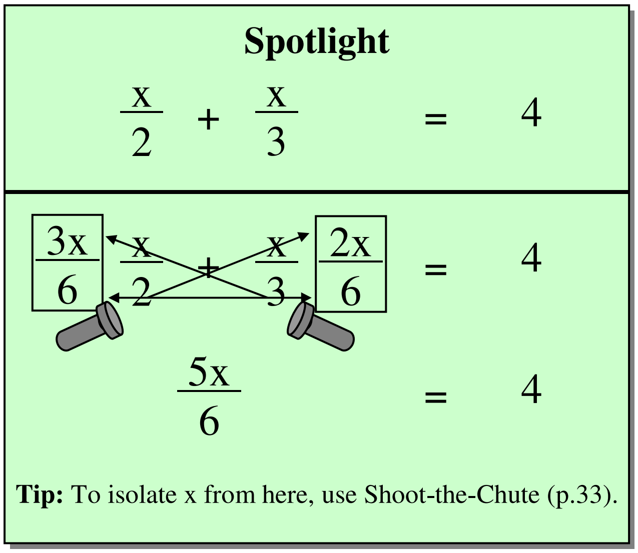
$$x = \frac{8}{3}$$

Your turn: Shoot-the-Chute to clear the fractions and isolate the variable.

$\frac{3}{5} = \frac{1}{x}$	$\frac{2x}{3} = \frac{3}{7}$	$\frac{5}{2} = \frac{4}{3x}$

Combining Fractions: Spotlighting!

As an alternative to clearing denominators by magnifying with the LCM (p.32), use the *spotlighting* technique to create equivalent fractions. (See *Max Learning's Fraction Fun: Xdm/Sh!*). If the original denominators are not prime numbers, factor them and "crush" any common factors before spotlighting.



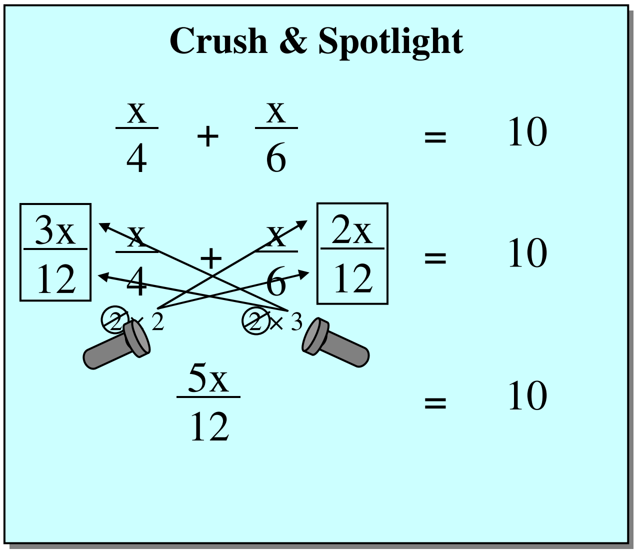
Spotlight

$$\frac{x}{2} + \frac{x}{3} = 4$$

$$\boxed{\frac{3x}{6}}\ \frac{x}{2} + \frac{x}{3}\ \boxed{\frac{2x}{6}} = 4$$

$$\frac{5x}{6} = 4$$

Tip: To isolate x from here, use Shoot-the-Chute (p.33).



Crush & Spotlight

$$\frac{x}{4} + \frac{x}{6} = 10$$

$$\boxed{\frac{3x}{12}}\ \frac{x}{4} + \frac{x}{6}\ \boxed{\frac{2x}{12}} = 10$$

$$\frac{5x}{12} = 10$$

Your turn: Spotlight to combine fractions.

$$\frac{x}{5} + \frac{x}{2} = 3$$

$$\frac{2x}{3} - \frac{x}{2} = 8$$

Your turn: Crush & spotlight to combine fractions.

$$\frac{2x}{3} + \frac{x}{6} = 7$$

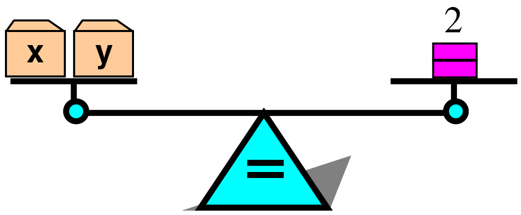
$$\frac{3x}{4} - \frac{5x}{8} = 9$$

Two Equations, Two Unknowns

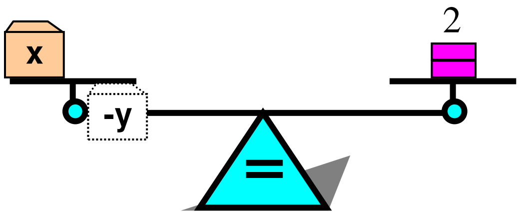
Some algebra problems involve Two Equations with Two first-power (x^1 & y^1) Unknowns, aka *simultaneous equations* or a *system of equations*. Their solution, if one exists, is the ordered pair (p.20) that satisfies both equations. For short, we'll call these 2EqUnk [ek-unk] problems.

Dilemma

If you have two equations each with two boxes (variables) on a scale, it's not obvious how to isolate either box to see what it contains.



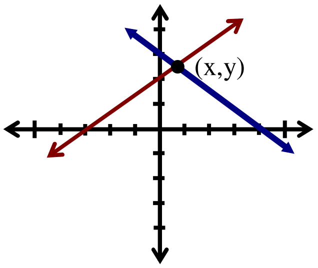
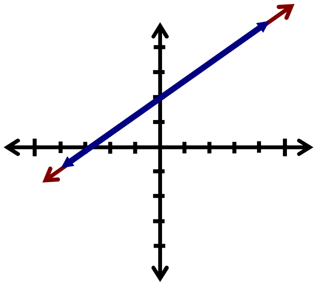
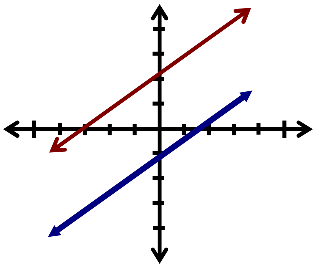
$x + y = 2$



$x - y = 2$

Remedy

Use Elimination (p.36) or Substitution (p.40) to transform the two equations into one equation with one unknown (1EqUnk p.25). Solve it, and use its solution to find the value of the second unknown variable.

<table>
<tr><th colspan="3">Possible 2EqUnk Outcomes
2EqUnk equations belong to the Linear Function Family (p.24), so each equation will graph as a sloped line.</th></tr>
<tr><td>Intersecting Lines
The solution is the point where the two lines cross</td><td>Identical (Collinear) Lines
The solution is every point on each line.</td><td>Parallel [PAIR-uh-lel] Lines
There is no solution, as parallel lines never touch.</td></tr>
<tr><td> </td><td></td><td></td></tr>
</table>

Spelling Tip

To spell "parallel," imagine you have a friend named El who likes to golf. To wish him luck, you say, "I hope you <u>par all El</u>!"

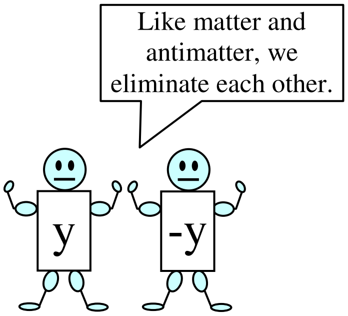
2EqUnk Elimination: You're outa here!

Use Elimination when both equations are in ax + by = c form.

1. Combine the 2EqUnks so as to eliminate *either* variable (pick the easier one).
2. Solve the resulting 1EqUnk to get the value of its variable.
3. Plug that value into either original equation, and solve for the eliminated variable.
4. Check the (x, y) solution in *both* original equations.

Add to Eliminate

Add equations that have matching, *oppositely*-signed variable terms.



Solve

$$\begin{array}{l} x + y = 2 \\ x - y = 2 \end{array}$$

1. Add to eliminate y

$$\begin{array}{rrcl} & x + \cancel{y} & = & 2 \\ + & \underline{x - \cancel{y}} & \underline{=} & \underline{2} \\ & 2x & = & 4 \end{array}$$

2. Solve 1EqUnk

$$\frac{\cancel{2}x}{\cancel{2}} = \frac{4}{2}$$

$$\boxed{x = 2}$$

x + y = 2

3. Plug & Chug

$$\begin{array}{rcl} x + y & = & 2 \\ \cancel{2} + y & = & 2 \\ \underline{\cancel{-2}} & & \underline{-2} \\ \boxed{y = 0} \end{array}$$

4. Check

$$\begin{array}{rcl} x + y & = & 2 \\ \mathbf{2} + \mathbf{0} & = & 2 \\ 2 & = & 2 \quad \surd \end{array}$$

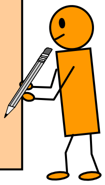
x − y = 2

3. Plug & Chug

$$\begin{array}{rcl} x - y & = & 2 \\ \cancel{2} - y & = & 2 \\ \underline{\cancel{-2}} & & \underline{-2} \\ (-1)(-y) & = & 0(-1) \\ \boxed{y = 0} \end{array}$$

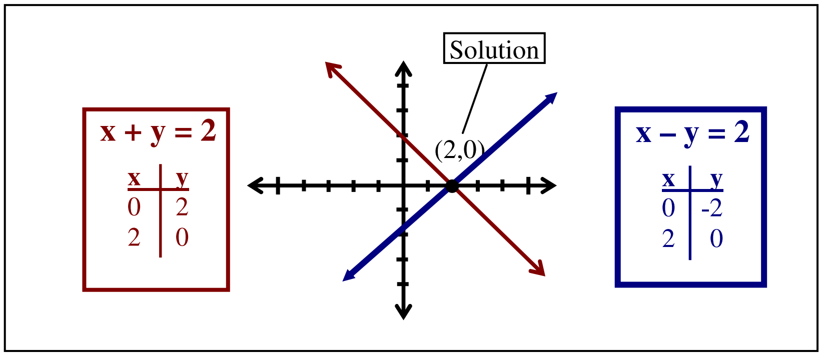
4. Check

$$\begin{array}{rcl} x - y & = & 2 \\ \mathbf{2} - \mathbf{0} & = & 2 \\ 2 & = & 2 \quad \surd \end{array}$$



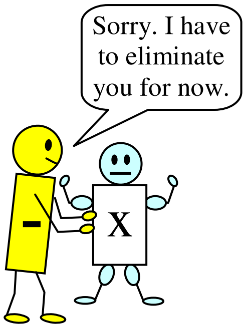
Solution
(x, y) = (2, 0)

It's sufficient to Plug & Chug (p.18) just one equation,
but always check both equations.



Subtract to Eliminate

Subtract equations that have matching, *same*-signed variable terms.



Solve

$$\begin{aligned} x + 2y &= 4 \\ x - y &= 1 \end{aligned}$$

1. Subtract to eliminate x

$$\begin{aligned} x + 2y &= 4 \\ -\underline{(x - y} &\underline{= 1)} \end{aligned}$$

$$\begin{aligned} \cancel{x} + 2y &= 4 \\ \underline{-\cancel{x} + y} &\underline{= -1} \\ 3y &= 3 \end{aligned}$$

2. Solve 1EqUnk

$$\frac{\cancel{3}y}{\cancel{3}} = \frac{3}{3}$$

$$\boxed{y = 1}$$

$x + 2y = 4$

3. Plug & Chug

$$\begin{aligned} x + 2y &= 4 \\ x + 2(\mathbf{1}) &= 4 \\ x + \cancel{2} &= 4 \\ \underline{-\cancel{2}} &\quad \underline{-2} \\ \boxed{x = 2} \end{aligned}$$

4. Check

$$\begin{aligned} x + 2y &= 4 \\ \mathbf{2} + 2(\mathbf{1}) &= 4 \\ 2 + 2 &= 4 \\ 4 &= 4 \ \surd \end{aligned}$$

$x - y = 1$

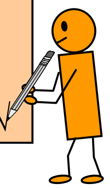
3. Plug & Chug

$$\begin{aligned} x - y &= 1 \\ x - \cancel{1} &= 1 \\ \underline{+\cancel{1}} &\quad \underline{+1} \\ \boxed{x = 2} \end{aligned}$$

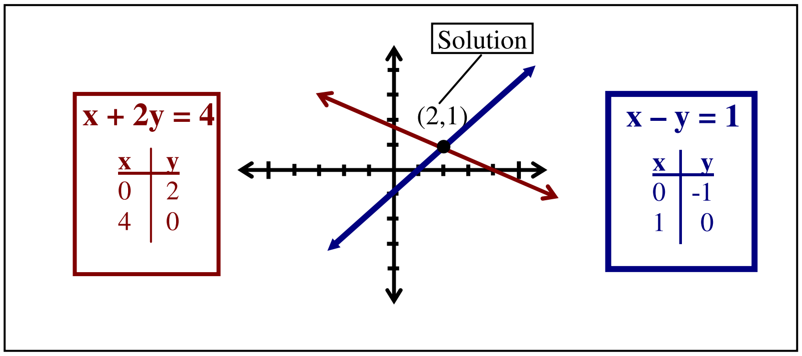
4. Check

$$\begin{aligned} x - y &= 1 \\ \mathbf{2} - \mathbf{1} &= 1 \\ 1 &= 1 \ \surd \end{aligned}$$

Solution
(x, y) = (2, 1)



It's sufficient to Plug & Chug (p.18) just one equation, but always check both equations.



Your turn: Add to eliminate and solve.

x + 2y = 5

3. Plug & Chug

4. Check

1. Add

$$\begin{array}{r} x + 2y = 5 \\ +\ \underline{x - 2y = 1} \end{array}$$

2. Solve 1EqUnk

x – 2y = 1

3. Plug & Chug

4. Check

Solution

(x, y) = (___, ___)

Your turn: Subtract to eliminate and solve.

2x + 2y = 6

3. Plug & Chug

4. Check

1. Subtract

$$\begin{array}{r} 2x + 2y = 6 \\ -\underline{(2x - y = 0)} \end{array}$$

2. Solve 1EqUnk

2x – y = 0

3. Plug & Chug

4. Check

Solution

(x, y) = (___, ___)

Multiply Then Eliminate

If the equations have no matching variable terms, multiply to create them.

No matching terms

$$\begin{aligned} 3x + 2y &= 4 \\ 2x - y &= 3 \end{aligned}$$

Multiply to eliminate y

$$\begin{aligned} 3x + 2y &= 4 \\ 2\,(2x - y &= 3) \end{aligned}$$

Eliminate

$$\begin{array}{rrcrcr} & 3x & + & \cancel{2}y & = & 4 \\ + & 4x & - & \cancel{2}y & = & 6 \\ \hline & 7x & & & = & 10 \end{array}$$

No matching terms

$$\begin{aligned} 2x + 3y &= 5 \\ x + 2y &= 2 \end{aligned}$$

Multiply to eliminate x

$$\begin{aligned} 2x + 3y &= 5 \\ -2\,(x + 2y &= 2) \end{aligned}$$

Eliminate

$$\begin{array}{rrcrcr} & \cancel{2}x & + & 3y & = & 5 \\ + & -\cancel{2x} & - & 4y & = & -4 \\ \hline & & & -y & = & 1 \end{array}$$

No matching terms

$$\begin{aligned} 3x + 4y &= 7 \\ -2x + 3y &= 1 \end{aligned}$$

Multiply to eliminate x

$$\begin{aligned} 2\,(3x + 4y &= 7) \\ 3\,(-2x + 3y &= 1) \end{aligned}$$

Eliminate

$$\begin{array}{rrcrcr} & \cancel{6}x & + & 8y & = & 14 \\ + & -\cancel{6x} & + & 9y & = & 3 \\ \hline & & & 17y & = & 17 \end{array}$$

Your turn: Multiply then eliminate.

$$\begin{aligned} 4x + 3y &= 5 \\ 2x - y &= 1 \end{aligned}$$

Multiply to eliminate y

Eliminate

$$\begin{aligned} 3x + 3y &= 1 \\ x + 2y &= 5 \end{aligned}$$

Multiply to eliminate x

Eliminate

$$\begin{aligned} 3x + 3y &= 8 \\ 2x - 2y &= 3 \end{aligned}$$

Multiply to eliminate y

Eliminate

In each case, you could choose to eliminate the opposite variable. The ultimate solution would be the same.

2EqUnk Substitution: Identical Twins

Use Substitution when one of the equations has an isolated x or y variable, or when one of the variables is relatively easy to isolate.

1. Substitute the isolated variable's equivalent expression into the other equation.
2. Solve the resulting 1EqUnk to get the value of the other variable.
3. Plug the other variable's value into the isolated variable's equation and solve.
4. Check the (x, y) solution in the other equation.

Solve

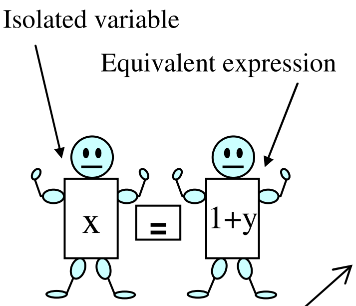
$$x = 1 + y$$
$$x + y = 5$$

1. Substitute

$$x + y = 5$$
$$(1+y) + y = 5$$

2. Solve 1EqUnk

$$\cancel{1} + 2y = 5$$
$$-\cancel{1} \qquad -1$$
$$\frac{\cancel{2}y}{\cancel{2}} = \frac{4}{2}$$
$$\boxed{y = 2}$$



BrainAid
Imagine one identical twin taking the place of the other and fooling the guests at a family reunion.

3. Plug & Chug

$$x = 1 + y$$
$$x = 1 + \mathbf{2}$$
$$\boxed{x = 3}$$

Solution
(x, y) = (3, 2)

4. Check

$$x + y = 5$$
$$\mathbf{3} + \mathbf{2} = 5$$
$$5 = 5 \quad \surd$$

Your turn: Substitute to solve.

Solve

$$y = 2 + x$$
$$x + y = 6$$

1. Substitute

2. Solve 1EqUnk

3. Plug & Chug

$$y = 2 + x$$

Solution
(x, y) = (__, __)

4. Check

$$x + y = 6$$

2EqUnk Identical or Parallel: All or Nothing

If elimination or substitution result in variable-less *equalities*, the lines produced are identical (aka collinear koh-LIN-ee-ur). This occurs when one equation is a multiple of the other.
EQUALITY = IDENTICAL

Solve

$$\begin{array}{rcl} 2x - 2y &=& 4 \\ x - y &=& 2 \end{array}$$

By Elimination

$$\begin{array}{rcl} 2x - 2y &=& 4 \\ \underline{-2(x - y} &\underline{=}& \underline{2)} \end{array}$$

$$\begin{array}{rcl} \cancel{2x} - \cancel{2y} &=& 4 \\ \underline{-\cancel{2x} + \cancel{2y}} &\underline{=}& \underline{-4} \\ 0 + 0 &=& 0 \end{array}$$

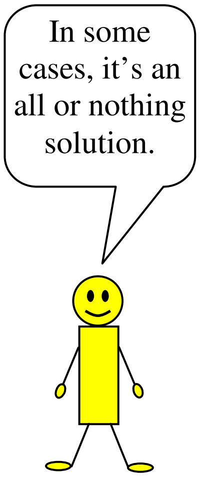
Variable-less Equality

By Substitution

$$\begin{array}{rcl} x - y &=& 2 \\ x &=& 2+y \end{array}$$

$$\begin{array}{rcl} 2x - 2y &=& 4 \\ 2\mathbf{(2+y)} - 2y &=& 4 \\ 4+\cancel{2y - 2y} &=& 4 \\ 4 &=& 4 \end{array}$$

Variable-less Equality



If elimination or substitution result in variable-less *inequalities*, the lines produced are parallel. This occurs when the equations have the same coefficients but different constants.
INEQUALITY = PARALLEL

Solve

$$\begin{array}{rcl} x - y &=& 1 \\ x - y &=& -1 \end{array}$$

By Elimination

$$\begin{array}{rcl} x - y &=& 1 \\ \underline{-(x - y} &\underline{=}& \underline{-1)} \end{array}$$

$$\begin{array}{rcl} \cancel{x} - \cancel{y} &=& 1 \\ \underline{-\cancel{x} + \cancel{y}} &\underline{=}& \underline{1} \\ 0 + 0 &<& 2 \end{array}$$

Variable-less Inequality

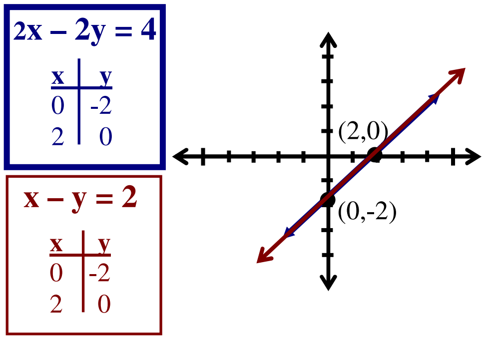
By Substitution

$$\begin{array}{rcl} x - y &=& 1 \\ x &=& 1+y \end{array}$$

$$\begin{array}{rcl} x - y &=& -1 \\ \mathbf{1+}\cancel{\mathbf{y} - y} &=& -1 \\ 1 &>& -1 \end{array}$$

Variable-less Inequality

Identical lines share *all* points.



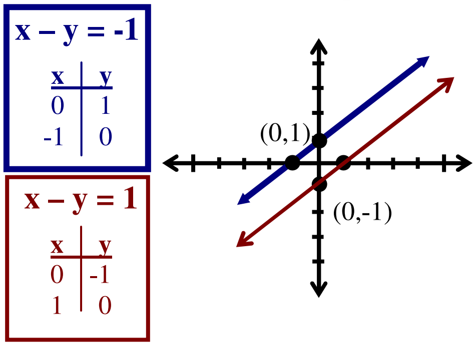
2x – 2y = 4

x	y
0	-2
2	0

x – y = 2

x	y
0	-2
2	0

Parallel lines share *no* points.



x – y = -1

x	y
0	1
-1	0

x – y = 1

x	y
0	-1
1	0

Linear Equations

Equations involving x^1 are called first-degree or Linear [LIHN-ee-ur] Equations.
When graphed, Linear Equations produce lines.
We'll call Linear Equations LinEqs [lin-eks] for short.

- Standard form for a one-variable LinEq is $ax + b = c$ (see 1EqUnk p.25).
- Standard form for a two-variable LinEq is $ax + by = c$ (see 2EqUnk p.35).
- Standard form for the Linear Function is: $f(x) = mx + b$ (see Function p.23).

Trap! The 'b' in $ax + by = c$ is different from the 'b' in $f(x) = mx + b$.

Slope-Intercept Form: y = mx + b

Slope-intercept form makes it easier to visualize and graph lines on Cartesian axes (p.20).
The y-coefficient must be +1. The x-coefficient 'm' is the slope. The constant 'b' is the y-intercept.
We can convert a standard two-variable LinEq to slope-intercept form by isolating the y variable.

Convert to slope-intercept form

$$\begin{array}{rcl} \cancel{2x} + 3y & = & 9 \\ \underline{\cancel{-2x}} \qquad & & \underline{-2x} \end{array}$$

$$\frac{\cancel{3}y}{\cancel{3}} = \frac{-2x + 9}{3}$$

$$\boxed{y = -\tfrac{2}{3}x + 3}$$

$m = -\tfrac{2}{3};\ b = 3$

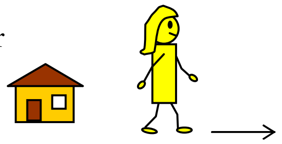
Your turn: Convert to slope-intercept form.

$-4x + 2y = 6$

Nature of LinEqs

LinEqs represent things that occur or change at a constant rate.

Problem: Starting 1 mile from home, Tia walks at a steady rate of 1 mile per hour towards the next town. How many miles from home is she after 2 hours?



Analysis: This is a Distance = Rate • Time (D=RT) travel problem (p.64) that fits neatly into slope-intercept form with y = Distance, m = Rate, x = Time, and b = starting point.

$y = mx + b$
$D = RT + b$

Plugging in the given values:

$D = 1\ \frac{\text{mile}}{\cancel{\text{hour}}} \cdot 2\ \cancel{\text{hours}} + 1\text{ mile}$

$D = 2\text{ miles} \qquad + 1\text{ mile}$

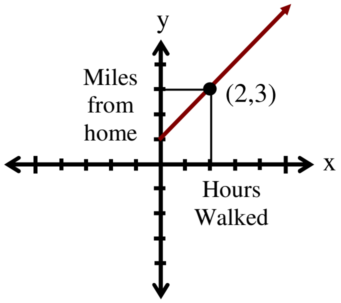
$D = \boxed{3\text{ miles}}$

Solution: Starting 1 mile from home, after 2 hours of walking, Tia was 3 miles from home.

This graph shows Tia's distance (miles) from home for the time (hours) that she walked.

$y = x + 1$

x	y
0	1
1	2
2	3
3	4



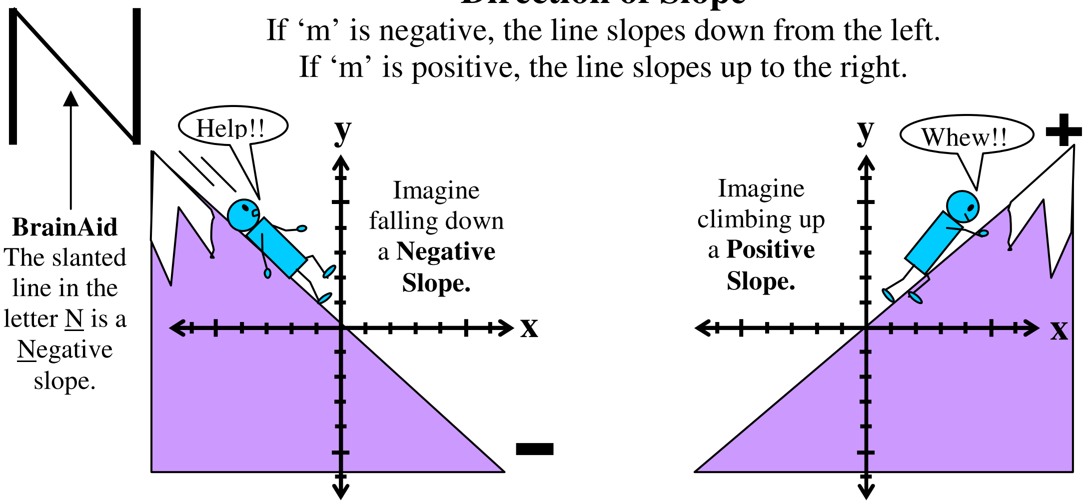
The usual solution to a LinEq problem is a single point, but the line produced by the LinEq displays all possible solutions, e.g., after 3 hours of walking, Tia was 4 miles from home.

Slope: m = mountain

In the LinEq y = mx + b, ‘m’ equals the slope.
The slope determines the direction and steepness of a line.

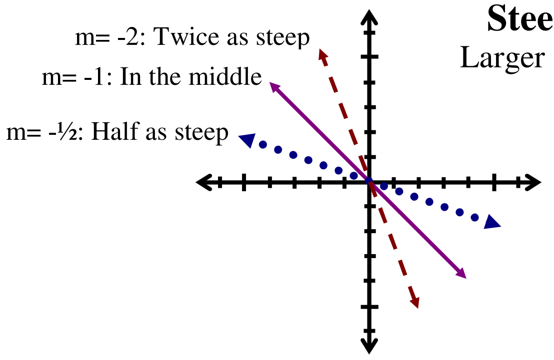
Direction of Slope

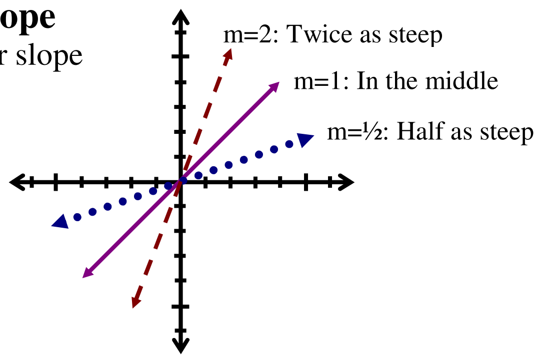
If ‘m’ is negative, the line slopes down from the left.
If ‘m’ is positive, the line slopes up to the right.



Steepness of Slope

Larger ‘m’ = Steeper slope

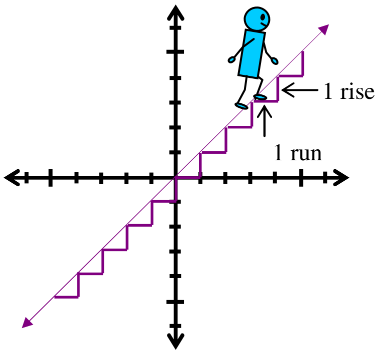
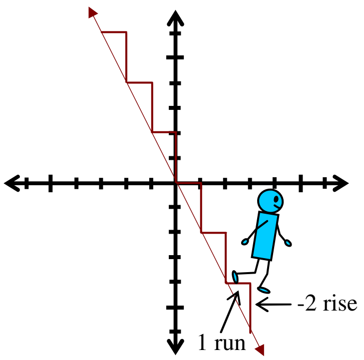
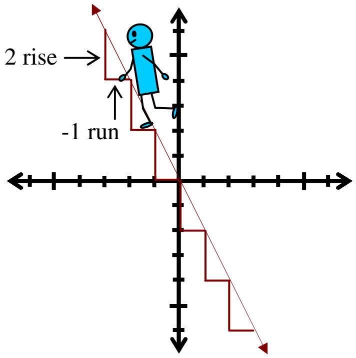




Arrowheads indicate that a line theoretically goes on forever in both directions.

Slope = Rise/Run

Slope is the ratio of a line’s *rise* (up/down) over its *run* (left/right).
BrainAid: Imagine steps that rise and run along the mountain slope to make it easier to climb or descend.

m = 1 1 rise / 1 run	**m = -2** -2 rise / 1 run	**m = -2** 2 rise / -1 run
		

Steps can be drawn below or above the slope line.

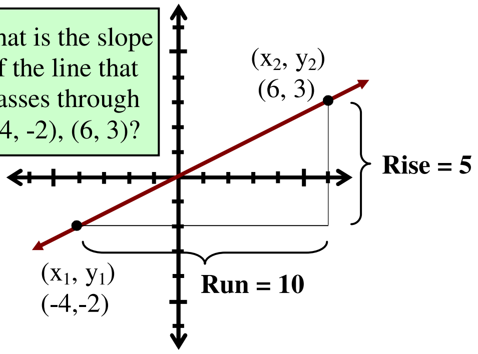
With a negative slope, either rise or run can be negative.

Calculating Slope: Δy / Δx

$$\frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{\text{change in y}}{\text{change in x}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The delta symbol Δ means "change in."
(x_2, y_2) and (x_1, y_1) represent any two points on a line.

What is the slope of the line that passes through (-4, -2), (6, 3)?



$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - -2}{6 - -4} = \frac{5}{10} = \enclose{circle}{\frac{1}{2}}$$

Traps!

Common Slope-Calculation Errors

Inconsistent subtraction direction

$$\frac{y_2 - y_1}{x_1 - x_2}$$

Inverted coordinates

$$\frac{x_2 - x_1}{y_2 - y_1}$$

Errors with negatives

-5 – -2 = -7
(should be -3)

Drop, Rotate, & Subtract

To minimize slope-calculation errors, drop y's down, rotate x's around, then subtract.

(x_2, y_2) (x_1, y_1)

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = m$$

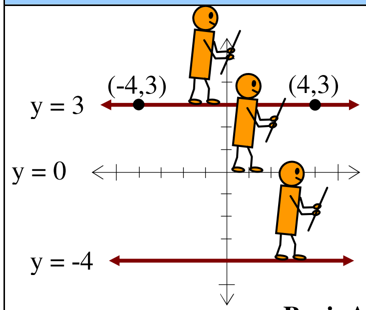
(-2, -1) (-6, 1)

$$\frac{-1 - 1}{-2 - -6} = \frac{-2}{4} = \frac{-1}{2}$$

Your turn: Drop, Rotate, & Subtract to find the slopes.

(3, 7) (2, 5)	(-3, 7) (-2, 5)	(-3, -7) (-2, -5)

Horizontal Line = Zero Slope



(4, 3) (-4, 3)

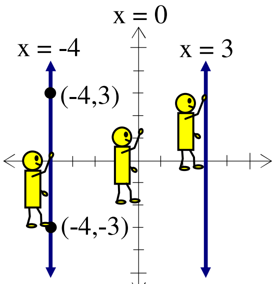
$$\frac{3 - 3}{4 - -4} = \frac{0}{8} = \mathbf{0}$$

BrainAid
Imagine walking on horizontal wires [y'urz].
Horizontal lines are functions (p.23).
Paradox: The equation for the x-axis is y = 0.

Vertical Line = Undefined Slope

(-4, 3) (-4,-3)

$$\frac{3 - -3}{-4 - -4} = \frac{6}{0} = ?$$



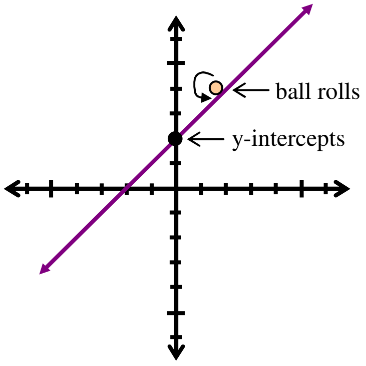
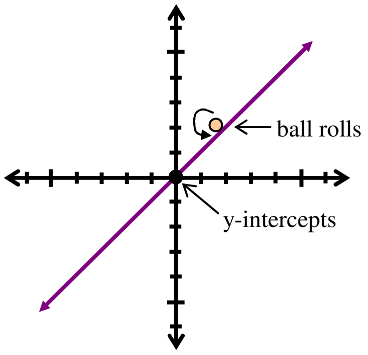
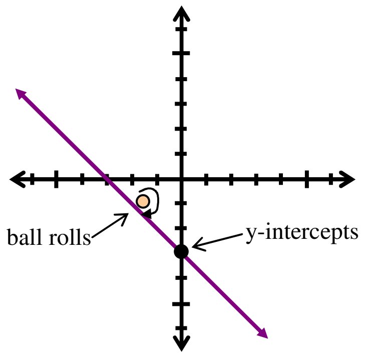
BrainAid
Imagine climbing eXtra high poles.
Vertical lines are *not* functions (p.23).
Paradox: The equation for the y-axis is x = 0.

Y-intercept: b = ball

In the LinEq y = mx + b, ‘b’ equals the y-intercept.
The y-intercept is the point where a line crosses the y-axis.
The y-intercept occurs when x = 0.

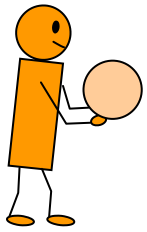
$$\begin{aligned} y &= mx + b \\ y &= m(\mathbf{0}) + b \\ \boxed{y} &\boxed{= \quad b} \end{aligned}$$

BrainAid: Imagine a ball rolling down a slope being intercepted by the y-axis.

y = x + 2 m = 1 ; b = 2	**y = x** m = 1 ; b = 0	**y = -x – 3** m = -1 ; b = -3
		

X-intercept: x = -b/m

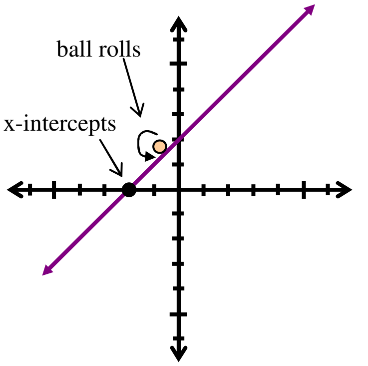
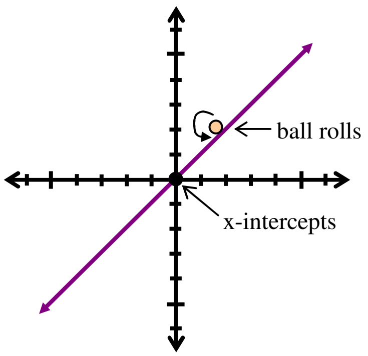
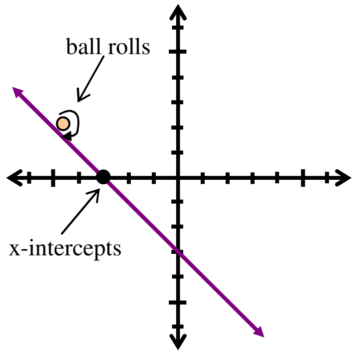
The x-intercept is the point where a line crosses the x-axis.
The x-intercept occurs when y = 0.



$$\begin{aligned} y &= mx + b \\ \mathbf{0} &= mx + \cancel{b} \\ \underline{-b} & \qquad\qquad \underline{\cancel{-b}} \\ \frac{-b}{m} &= \frac{\cancel{m}x}{\cancel{m}} \\ \boxed{\frac{-b}{m}} &\boxed{= x} \end{aligned}$$

BrainAid
Imagine the x-axis
intercepting a negative
ball (-b) over a
mountain (-b/m).

BrainAid: Imagine a ball rolling down a slope being intercepted by the x-axis.

y = x + 2 m = 1; b = 2 -b/m = -2/1 = -2	**y = x** m = 1 ; b = 0 -b/m = -0/1 = 0	**y = -x – 3** m = -1 ; b = -3 -b/m = --3/-1 = -3
		

Plotting LinEqs

It takes a minimum of two points to define a line. You have several plotting options.

Plot using y-intercept and slope

Use with slope-intercept $y=mx+b$ form.

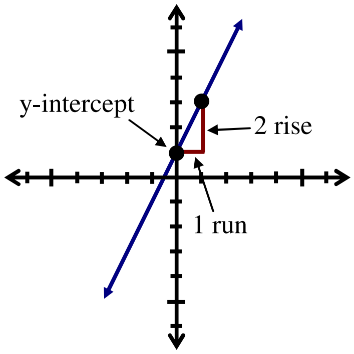
1. Draw the y-intercept point on the y-axis.
2. From that point, follow rise/run to the 2nd point.

Your turn: Plot using y-intercept and slope.

$y = 2x+1$

$b = 1$

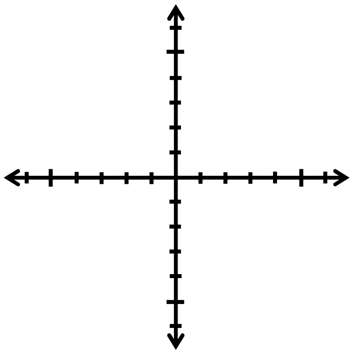
$m = 2/1$



$y = 3x–2$

$b =$ ___

$m =$ ___



Plot using x & y intercepts

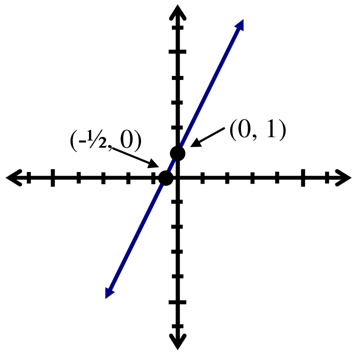
Use with standard $ax+by=c$ form.

1. Set $x=0$, solve for y, plot the y-intercept.
2. Set $y=0$, solve for x, plot the x-intercept.

Your turn: Plot using x & y intercepts.

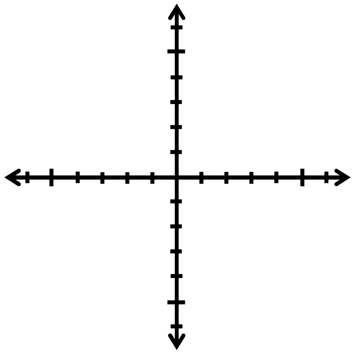
$-2x+y = 1$

x	y
0	1
-½	0



$-3x+y = -2$

x	y



Plot using x & y coordinates

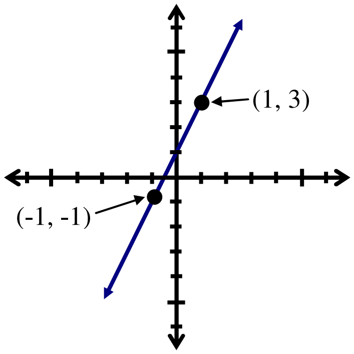
Alternate for standard $ax+by=c$ form.

1. Choose any x, solve for y, plot the point.
2. Choose any 2nd x, solve for y, plot the point.

Your turn: Plot using x & y coordinates.

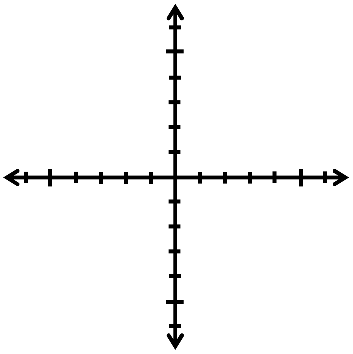
$-2x+y = 1$

x	y
-1	-1
1	3



$-3x+y = -2$

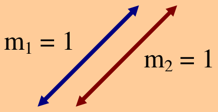
x	y

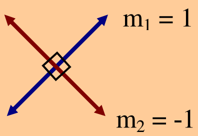


Parallel & Perpendicular Line Slopes

Given two separate lines with slopes m_1 and m_2:

- If $m_1 = m_2$, the lines are parallel (never touch).
- If $m_1 \bullet m_2 = -1$, the lines are perpendicular [pur-pen-DIH-kyu-lur] (cross at 90° angles).





Quadratic Equations

Equations involving x^2 are called second-degree or Quadratic [kwaw-DRA-tik] Equations.
When graphed, Quadratic Equations produce bowl-shaped parabolas (p.24).
We'll call Quadratic Equations QuadEqs [kwaw-deks] for short.

The Quadratic Function is $f(x) = ax^2+bx+c$ (see Function p.23).
ax^2 = quadratic term, bx = linear term, c = constant term

Remember that f(x) is the same as y.

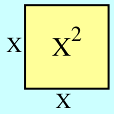
Standard Form: $ax^2 + bx + c = 0$

A standard QuadEq is a special case of the Quadratic Function where $f(x) = 0$.
Traditionally, the 0 is moved to the right side of the equation.

A standard QuadEq is a trinomial (p.14), but it can be a binomial or monomial as follows:

If $c=0$: $ax^2 + bx = 0$ If $b=0$: $ax^2 + c = 0$ If $b=0$ & $c=0$: $ax^2 = 0$

Question: Since "quad" implies "four," why don't QuadEqs involve x^4 instead of x^2?
Answer: "Quad" comes from the Latin "quadrate" which means "squared numbers."
Also, "quadrus" means "square." FYI: Equations with x^4 are called Quartic Equations.



A square has 4 sides.

Nature of QuadEqs

QuadEqs represent things that occur or change at variable rates.

QuadEq Example

Through observation and experiment, scientists devised a quadratic equation that gives the height (at any time during its flight) of an object shot or thrown straight up into the air.
They named it the Position Function.

$h(t) = -16t^2 + vt + h$

h(t) = height (feet) as a function of time in flight
–16 = gravitational pull (feet/second per second)
t = time (seconds)
v = initial velocity (feet/second)
h = initial height above ground (feet)

Problem: A cannonball is shot straight up from the ground. Its initial velocity is 160 feet/second, but it's slowed by the pull of gravity, stops, reverses direction, and returns to earth. How long was its flight?

Analysis: The cannonball starts on the ground, so its initial height h is 0 feet. When it lands after t seconds, its height h(t) as a function of time is also 0 feet.

If we reverse the Position Function, the problem neatly fits into a standard form QuadEq with a= -16, $x^2=t^2$, b=v, x=t, c=h:

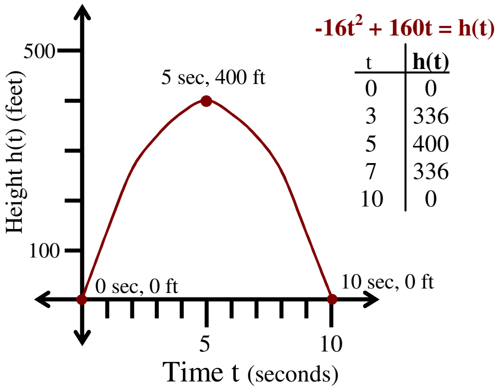
$$\begin{array}{lllllll}
\mathbf{ax^2} & \mathbf{+} & \mathbf{bx} & \mathbf{+} & \mathbf{c} & \mathbf{=} & \mathbf{0} \\
-16t^2 & + & vt & + & h & = & h(t) \\
-16t^2 & + & 160t & + & 0 & = & 0 \\
\multicolumn{5}{l}{-16t\,(t - 10)} & = & 0
\end{array}$$

Factoring out -16t.

$t = 0$ or $t = 10$ Either t makes the equation equal 0.

Solution: The ball is launched from the ground at t=0 seconds and returns to the ground when t=10 seconds.

Although it may look like the ball's flight path, this graph shows a "time" path. The cannonball went straight up and down.



t	h(t)
0	0
3	336
5	400
7	336
10	0

The usual solution to a QuadEq is one or both x-intercepts, where $f(x) = 0$. However, the graph shows *all* heights at any time during flight, e.g., at 5 seconds, the cannonball reached a maximum height of 400 ft.

Analyzing Coefficients: Easy as a-b-c

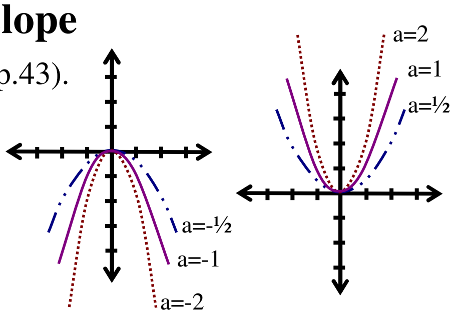
Knowing the effects of each coefficient (p.14) can make it easier to visualize and graph QuadEqs.

$y = ax^2$: almost like slope

‘a’ sets direction and steepness (like the slope ‘m’ in a LinEq p.43).

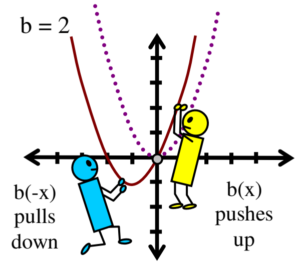
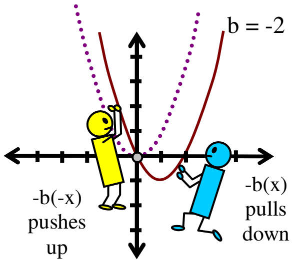
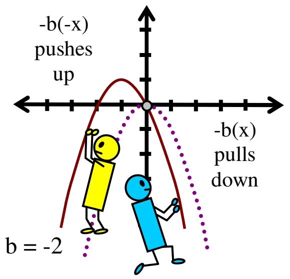
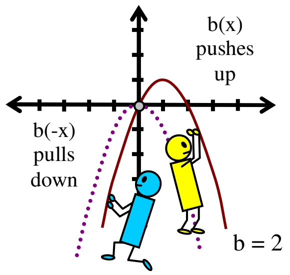
- Negative ‘a’ creates an inverted parabola (bowl down).
- Positive ‘a’ creates an upright parabola (bowl up).

- Larger ‘a’ creates steeper sides (narrower bowl).
- Smaller ‘a’ creates flatter sides (wider bowl).



$y = ax^2+bx$: bowl over

‘b’moves the parabola’s bowl up or down & left or right.
At x = 0, ‘b’ has no effect, so the y-intercept becomes the pivot point.

Upright Bowl Moves Down		Inverted Bowl Moves Up	
+b moves bowl left	**-b** moves bowl right	**–b** moves bowl left	**+b** moves bowl right
			

$y = ax^2+bx+c$: intercept

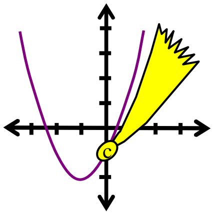
‘c’ is the y-intercept—where the parabola crosses the y-axis.

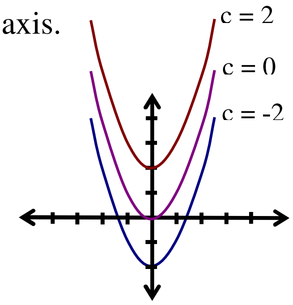
When x = 0:

$y = a(0)^2+b(0)+c$

$\mathbf{y = c}$

BrainAid
Imagine a comet being
intercepted by the y-axis.



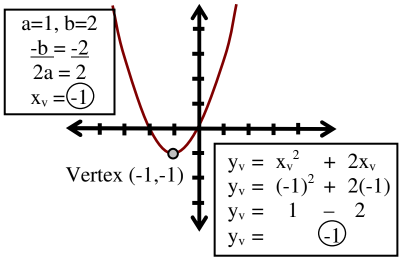


-b/2a: x-vertex

-b/2a is the x-coordinate (x_v) of the vertex—which is exactly halfway between the x-intercepts.
To find the y-coordinate (y_v) of the vertex, substitute (-b/2a) for x and solve.

Upright Parabola Vertex = bottommost or minimum point	**Inverted Parabola** Vertex = topmost or maximum point

$y = x^2+2x$



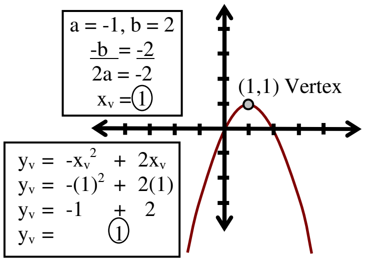
a=1, b=2

$\frac{-b}{2a} = \frac{-2}{2}$

$x_v = \enclose{circle}{-1}$

$y_v = x_v^2 + 2x_v$
$y_v = (-1)^2 + 2(-1)$
$y_v = 1 - 2$
$y_v = \enclose{circle}{-1}$

$y = -x^2+2x$



a = -1, b = 2

$\frac{-b}{2a} = \frac{-2}{-2}$

$x_v = ①$

$y_v = -x_v^2 + 2x_v$
$y_v = -(1)^2 + 2(1)$
$y_v = -1 + 2$
$y_v = ①$

Multiplying & Factoring Expressions

Multiplying and factoring are opposite operations.

Traditional Techniques

Multiply Monomial • Binomial

$x\ (x + 2)$	Distribute x over (x + 2)
$x^2 + 2x$	Result: binomial

Factor Binomial

$x^2 + 2x$	Extract GCF (p.10) of x
$x\ (x + 2)$	Result: monomial • binomial

Multiply Binomial • Binomial

$(x + 1)\ (x + 2)$	Distribute x over (x + 2) Distribute 1 over (x + 2)
First Outside Inside Last $x^2 + 2x + x + 2$	Combine 'x' terms.
$x^2 + \quad 3x \quad + 2$	Result: Trinomial

This is commonly called the FOIL method for its distribution order: First, Outside, Inside, Last.

Factor Trinomial

$x^2 + 3x + 2$	Factor First term
$(x \quad)\ (x \quad)$	Factor Last term
$(x + 1)\ (x + 2)$	Result: binomial • binomial

This method usually requires trial and error until the Outside/Inside products combine to produce the middle term.

Multiply + and – Binomials

$(x + 2)\ (x - 2)$	Distribute x over (x – 2) Distribute 2 over (x – 2)
$x^2 - 2x + 2x - 4$	Combine x terms.
$x^2 \qquad\qquad - 4$	Result: Difference of 2 Squares

Factor Difference of 2 Squares

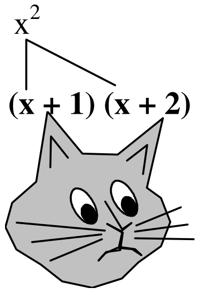
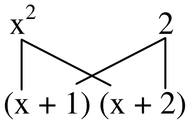
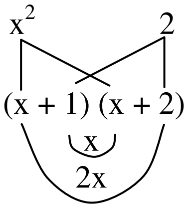
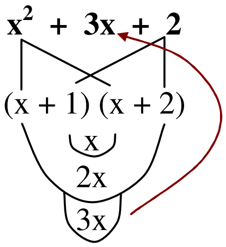
$x^2 - 4$	Factor 1st term
$(x \quad)\ (x \quad)$	Factor 2nd term
$(x + 2)\ (x - 2)$	Result: + and – binomials

Your turn: Multiply or factor the following expressions.

Multiply $x\ (x + 3)$ Factor $x^2 + 3x$	Multiply $(x + 2)\ (x + 3)$	Multiply $(x + 3)\ (x - 3)$
Multiply $2x\ (x + 3)$ Factor $2x^2 + 6x$	Factor $x^2 + 5x + 6$ **Tip:** Factor 6 into 2 • 3.	Factor $x^2 - 9$

Cat Techniques

Here are some fun and memorable way to multiply binomials and factor trinomials.

<table>
<tr><th colspan="4">Multiply Binomial “Eyes” into Trinomial Expression</th></tr>
<tr><th>Raise 1st Ear
Multiply first terms</th><th>Raise 2nd Ear
Multiply last terms</th><th>Make Nose & Mouth
Multiply inner/outer terms</th><th>Drop Tongue
Add inner/outer products</th></tr>
<tr>
<td> </td>
<td> </td>
<td> </td>
<td>
Flick middle term to top.</td>
</tr>
</table>

Your turn: Multiply the cat’s binomial eyes to create its face and a trinomial expression.

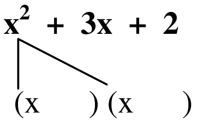
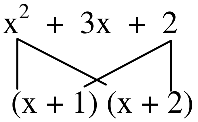
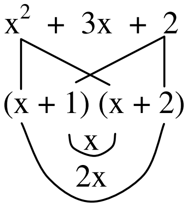
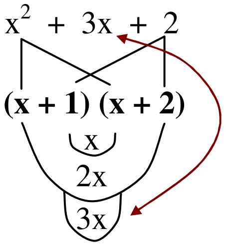
Raise 1st Ear
Multiply first terms

Raise 2nd Ear
Multiply last terms

$$(x + 2)(x + 3)$$

Make Nose & Mouth
Multiply inner/outer terms

Drop Tongue
Add inner/outer products

<table>
<tr><th colspan="4">Factor Trinomial Expression into Binomial “Eyes”</th></tr>
<tr><th>Drop 1st Ear
Factor first term</th><th>Drop 2nd Ear
Factor last term</th><th>Nose & Mouth Check
Multiply inner/outer terms</th><th>Tongue Taste Test
Add inner/outer products</th></tr>
<tr>
<td> </td>
<td> </td>
<td> </td>
<td>
Yum!
Middle terms match!</td>
</tr>
</table>

Your turn: Factor the trinomial expression to create a cat’s face with binomial eyes.

$$x^2 + 5x + 6$$

Drop 1st Ear
Factor first term

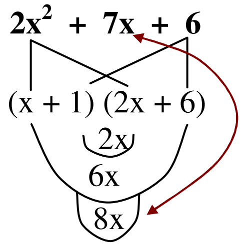
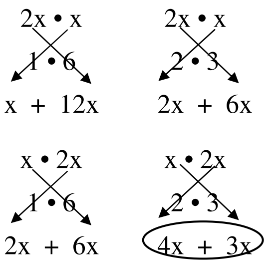
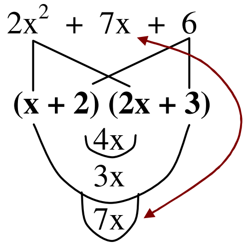
Drop 2nd Ear
Factor last term

Nose & Mouth Check
Multiply inner/outer terms

Tongue Taste Test
Add inner/outer products

Cat Traps & Tips

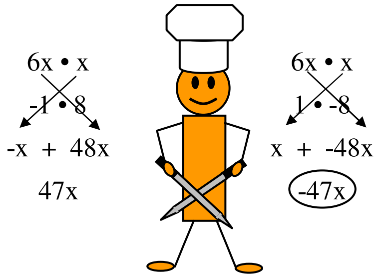
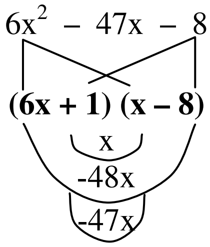
Factoring Trap: Cat Won't Eat!

Cat won't eat! Sum of inner/outer products doesn't match middle term.	**List Ingredients** List all possible factors for first and last terms.	**Prepare Food** Combine sets of factors, cross multiply, and add.	**Feed Cat** Use a food combination that matches the middle term.
 **Yuch!**	$2x^2 + 7x + 6$ 2x • x 1 • 6 x • 2x 2 • 3 List first term's factors forwards and backwards so cover all combinations. List in 1, 2, 3, 4.... order so don't overlook factors.	 In this case, only the fourth combination adds to 7x.	 **Yum!**

Your turn: Factor the trinomial and feed the cat.

List Ingredients	**Prepare Food**	**Feed Cat**
$3x^2 - 8x + 4$ **Tip:** The factors of +4 must be negative to get a –8 in the middle.		

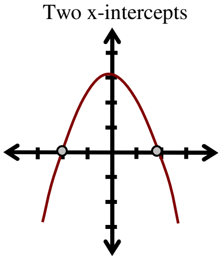
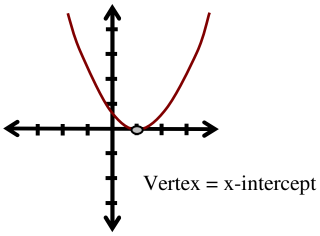
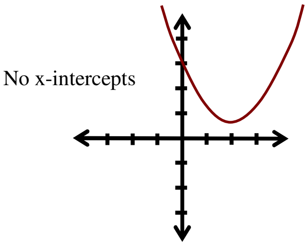
Factoring Tip: Analyze the Food!

List Ingredients List all possible factors for first and last terms.	**Prepare Food** Analyze the middle term to narrow down combinations. Cross multiply and add to find acceptable food.	**Feed Cat** Use a food combination that matches the middle term.
$6x^2 - 47x - 8$ x • 6x -1 • 8 6x •x 1 • -8 2x • 3x -2 • 4 3x • 2x 2 • -4 List forwards and backwards. List +/– combos.	**Analysis:** This problem has 4 • 4 = 16 combinations!! But the large middle term -47x suggests we first test combinations that multiply our largest factors 6 and 8.  Luckily, it took only two tries to get the right food!	 **Yum!**

Solving QuadEqs

To solve a QuadEq, put it in standard form and find its x-intercept/s (aka root/s).
Standard form sets the QuadEq to zero: $ax^2 + bx + c = 0$.
The value/s of x that make $f(x) = 0$ are the x-intercepts (x across, zero high/low).

If the parabola produced by a QuadEq touches the x-axis, the solutions are real numbers (p.6)
If the parabola does *not* touch the x-axis, the solutions are imaginary numbers (p.55).

Two real-number solutions	One real-number solution	Imaginary-number solution/s
		

Zero-Product Principle

If a product is zero, at least one of its factors must be zero.

If a • b = 0	If (x)(x+1) = 0	If (x–1)(2x+1) = 0
then	then	then
$\boxed{a = 0}$	$\boxed{x = 0}$	$x - 1 = 0$ $\boxed{x \quad = 1}$
and/or	and/or	and/or
$\boxed{b = 0}$	$x + 1 = 0$ $\boxed{x \quad = -1}$	$2x + 1 = 0$ $\boxed{x \quad = -½}$

Your turn: Apply the Zero-Product Principle to solve for the value/s of x.

$2x (x + 3) = 0$	$(x – 3) (x + 2) = 0$	$(2x – 1) (3x – 6) = 0$

Solving QuadEqs by Cat Factoring

Factor to Solve $x^2 + 7x + 12 = 0$

List Ingredients

$\underline{x^2} + 7x + \underline{12} = 0$

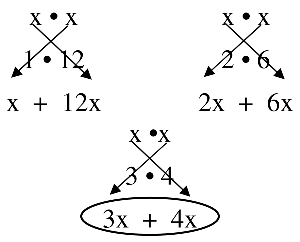
$x \cdot x$

$1 \cdot 12$

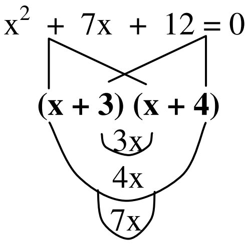
$2 \cdot 6$

$3 \cdot 4$

Prepare Food



Feed Cat



Yum!

Apply Zero-Product Principle

$(x + 3)(x + 4) = 0$

$x + 3 = 0$

$\boxed{x = -3}$

$x + 4 = 0$

$\boxed{x = -4}$

Check Solution/s

$x^2 + 7x + 12 = 0$

$(-3)^2 + 7(-3) + 12 = 0$

$9 - 21 + 12 = 0$

$-12 + 12 = 0$

$0 = 0$ √

$x^2 + 7x + 12 = 0$

$(-4)^2 + 7(-4) + 12 = 0$

$16 - 28 + 12 = 0$

$-12 + 12 = 0$

$0 = 0$ √

Your turn: Factor and solve.

List Ingredients	Prepare Food	Feed Cat
$x^2 + 6x + 9 = 0$		

Apply Zero-Product Principle	Check Solution/s
Tip: There is only one solution for x.	

Solving QuadEqs with Quadratic Formula

A QuadEq that can't be factored is called *prime*.
Use the Quadratic Formula to discover its x-intercepts.

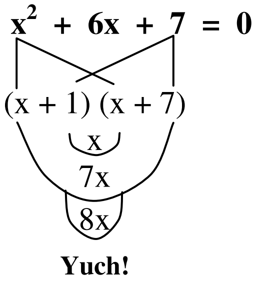
$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

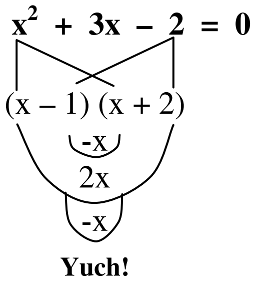
This complicated-looking formula was derived from $ax^2 + bx + c = 0$ using a process called Completing the Square. It looks scary, but it's simple to use: Substitute the values of the coefficients a, b, and c, then evaluate the expression. The result will be the x-intercept/s.

Discriminant [di-SKRI-mi-nunt]: $b^2 - 4ac$

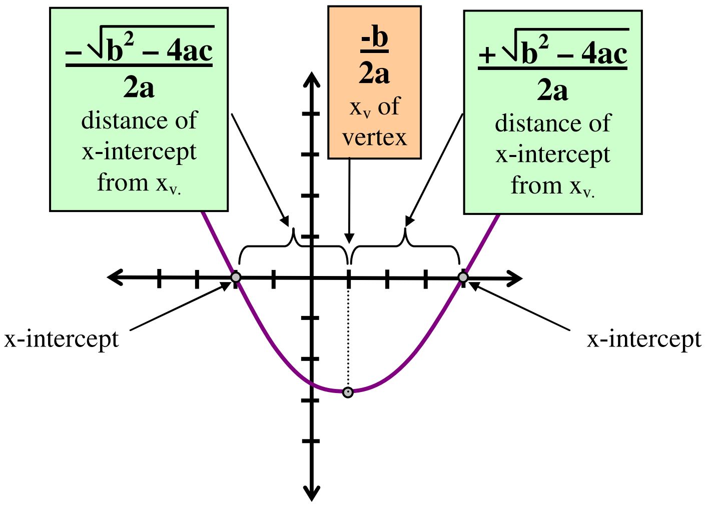
If the discriminant (the expression inside the square root radical sign) evaluates to a:

- Perfect square (e.g., 0, 1, 4, 9, 16...)—Solutions are rational real numbers (p.6).
- Positive number (e.g., 2, 3, 5, 6...)—Solutions are irrational real numbers (p.6).
- Negative number (e.g., -1, -2, -3, -4...)—Solutions are imaginary/complex numbers (p.55).

When factoring won't work...	**...use the Quadratic Formula**
 **Tip** Always write out the coefficient values, paying special attention to + or – signs.	$a = 1, b = 6, c = 7$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)}$ $x = \frac{-6 \pm \sqrt{36 - 28}}{2} \longrightarrow \sqrt{8}$ $\sqrt{4(2)}$ $x = \frac{-6 \pm 2\sqrt{2}}{2} \longleftarrow 2\sqrt{2}$ $\boxed{x = -3 - \sqrt{2}}$ $\boxed{x = -3 + \sqrt{2}}$ The solutions are irrational real numbers.

Your turn: When factoring won't work...	...use the Quadratic Formula
	a = ___, b = ___, c = ___ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Components of the Quadratic Formula



Imaginary/Complex Number Solutions

If the discriminant evaluates to a negative number, the QuadEq has an imaginary number solution.

Imaginary Number *i*

When first encountered, $\sqrt{-1}$ was thought to be an impossibility, because squaring a root was always thought to produce a positive square, e.g., 1 • 1 = +1 and –1 • –1 = +1.

And yet, $\sqrt{-1} \bullet \sqrt{-1} = -1$.

So, to contrast it with the real numbers (rational and irrational), $\sqrt{-1}$ was dubbed an “imaginary” number and represented by the italicized variable *i*.

$$i = \sqrt{-1}$$

In a sense, all numbers are “imaginary” because they only *represent* what is real. But in fact, *i* does represent real phenomena that occur in nature, particularly in the area of subatomic particles.

Complex Number

A complex number consists of a real number and an imaginary number. Example: *3 + i*

$$\mathbf{3x^2 + 4x + 2 = 0}$$

a = 3, b = 4, c = 2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(2)}}{2(3)}$$

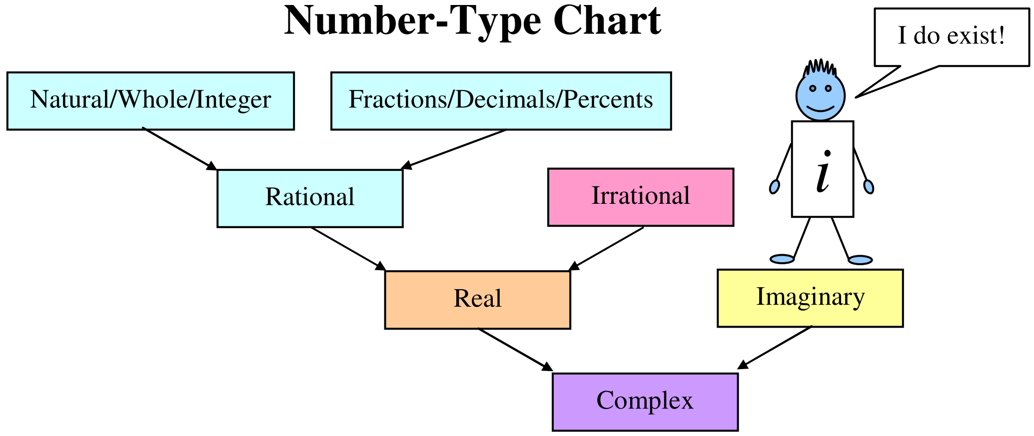
$$x = \frac{-4 \pm \sqrt{16 - 24}}{6} \longrightarrow \sqrt{-8}$$

$$\sqrt{4(-2)}$$

$$x = \frac{-4 \pm 2\sqrt{2}\,\sqrt{-1}}{6} \longleftarrow 2\sqrt{-2}$$

$$\boxed{\frac{-2 \pm i\sqrt{2}}{3}}$$

Number-Type Chart

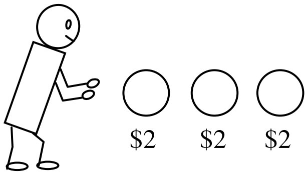


Word Problems

Of all areas of math, word problems (aka story problems) cause the most headaches. Why? Because they're written in words! It's sometimes tough to translate imprecise English words into precise math symbols. For all the anxiety they cause, I sometimes call word problems "worry" problems. But if you like to solve puzzles, this is where the fun begins!

Word Problem IDEAS

Use IDEAS to Identify/Draw/Equate/Assign/Solve word problems.

IDEAS	Explanation	Example
Identify	**Identify** the problem type. Nothing will aid you more in finding a solution. See Word Problem Types (p.58) for a list and references to page numbers with examples.	**Problem:** How much did Sam pay for three $2 beach balls? **Type:** Cost problem CPK (p.71)
Draw	**Draw** simple pictures or symbols of the items in the problem. Label values and units of measure. This will help you "see" beyond the words, which can be confusing.	
Equate	**Equate** the given and unknown values into a "word" equation. Use the English-to-Math Chart (p.57) as needed. Underline sets of words that represent values.	Cost paid equals price for one ball times the quantity of balls.
Assign	**Assign** a variable to each set of underlined words in the "word" equation. Predefined equations may use specific variables.	C = PK
Solve	**Solve** for the unknown variable/s by plugging in given values, including units. Keep items vertically aligned. Circle the answer/s. Unit Analysis: Make sure the units of measure work out appropriately (p.60). Convert Units: As needed (p.75). Check: Plug values back into the equation/s to verify your answer/s.	$C = \$2/\cancel{\text{ball}}\ (3\ \cancel{\text{balls}})$ $C = \boxed{\$6}$ **Check** C = PK 6 = 2(3) 6 = 6 √

Although you can probably solve most of the purposely-simple demonstration problems that follow without doing so, take time to complete each of the IDEAS steps, so that you'll be prepared to set up and solve more complex problems you may encounter in the future.

English-to-Math Chart

This chart lists words used in word problems and their math equivalents.
Add more examples to the chart as you encounter them.

One of the major hurdles you'll encounter in word problems is the tremendous number of ways that the same thing can be said with different words. And sometimes the same word can have different meanings. For example, the word "of" can mean either multiplication or division depending on how it is used.

ENGLISH	MATH	Sample Sentences	Equation
➢ equal ➢ is ➢ are ➢ has ➢ had	=	❖ Ann is the same age as Bob. ❖ Ann and Bob are equal in height. ❖ Ann has as many items as Bob.	A = B
➢ add ➢ sum ➢ plus ➢ more ➢ greater ➢ older ➢ increased by	+	❖ Cal has 3 more items than Deb. ❖ Cal is 3 years older than Deb. ❖ Cal's share increased by 3 over Deb's.	C = D + 3
➢ subtract ➢ difference ➢ minus ➢ less ➢ fewer ➢ younger ➢ remainder ➢ left	–	❖ Earl has 4 items fewer than Fran. ❖ Earl is 4 years younger than Fran. ❖ Earl got what was left after Fran used 4.	E = F – 4
➢ multiply ➢ product ➢ times ➢ times as many as ➢ @ (at) ➢ increased by a factor of	•	❖ Gene has 5 times what Hal has. ❖ Gene has 5 times as many as Hal. ❖ Gene bought 5 items @ $H each.	G = 5H
➢ divide ➢ quotient ➢ split ➢ per ➢ reduced by a factor of	/	❖ Ida's share was Jo's share divided by 6. ❖ Ida's share equals Jo's split 6 ways. ❖ Ida equals Jo's reduced by a factor of 6.	I = J/6
➢ fraction *of*	•	❖ Gene has half of what Hal has.	G = ½H
➢ whole number *of*	/	❖ Ken has 2 of 3 items.	K = 2/3

Word Problem Types

Many word problems use predefined equations that are based on patterns discovered in nature, math, or science. Below are some of the more common equation patterns and their problem types.

Q=RK (kyu-rik) Problems

Q: Quantity
R: Rate of change of Q/K
K: Kwantity (made-up word)

$$Q = R\,K$$
$$Q = \frac{Q}{\cancel{K}}\,\cancel{K}$$
$$Q = Q$$

K units dissolve, leaving only Q units

Alternate equations
R=Q/K; K=Q/R

Travel Problems
D=RT (p.64)
Distance = Rate of travel • Time
(Distance = Distance/Time • Time)

D=MV (p.67)
Distance = Mileage rate • Volume
(Miles = Miles/Gallon • Gallons)

Cost Problems
C=PK (p.71)
Cost = Price rate • Kwantity
(Cost = Price/Unit • Units)

Work Problems
W=RT (p.73)
Work = Rate of work • Time
(Work = Work/Time • Time)

Coin Problems
T=VC (p.72)
Total value = Value of coin • Coin quantity
(Value = Value/Coin • Coins)

Conversion Problems
N=CO (p.75)
New units = Conversion Rate • Old units
(New = New/Old • Old)

Physical Problems
W=EI†
Weight = Each's weight • Items
(Weight = Weight/Item • Items)

M=DV†
Mass = Density • Volume
(Mass = Mass/Volume • Volume)

V=FT†
Volume = Fill rate • Time
(Volume = Volume/Time • Time)

Trap!

Textbooks use a wide variety of variables for predefined equations. Often the same variable is used to represent different items, e.g., 'P' can represent Percent, Price, Perimeter, Principal, etc.; 'R' can represent various rates, like speed, work, or percent.

Equation BrainAids

Most variables on this page were chosen and arranged to make it easier to remember the equations.

See the individual BrainAids on the referenced pages.

Tip

As you encounter other problem types, add them to this page, or insert an additional sheet of paper to record them.

Q=PK (kyu-pik) Problems

Q: Quantity
P: Percent
K: Kwantity (made-up word)

P has no units, so Q & K have the same units.

Alternate equations
P=Q/K; K=Q/P

Interest Problems
I=RP (p.72)
Interest = Rate of return • Principal
($ Income = Percent • $ Invested)

Mixture Problems
V=AT (p.74)
Volume = Amount • Total
($\text{Volume}_{\text{part}}$ = Percent • $\text{Volume}_{\text{Total}}$)

$Q=K_1K_2$ (kyu-kik) Problems

Q: Quantity
K_1: Kwantity 1
K_2: Kwantity 2

If K_1, K_2 use same units, $Q=\text{units}^2$
If K_1, K_2 use different units, $Q=\text{unit}_1 \bullet \text{unit}_2$

Alternate equations
$K_1=Q/K_2$; $K_2=Q/K_1$

Area of Rectangle
A=WL (p.63)
(Area = Width • Length)

Electrical Power
E=KH†
(Energy = Kilowatts • Hours)

Other Types

Freeform Problems
1EqUnk/2EqUnk (p.61)

Markup Problems
N=O+MO (p.68)
New = Old + Markup% • Old

Discount Problems
N=O-DO (p.69)
New = Old – Discount% • Old

Percent-Change Problems
P = (N-O)/O (p.70)
Percent-change = (New – Old) / Old

† Problem types without page numbers are listed here for your use, but no examples follow.

Word Problem Analysis

To be solvable, a word problem must either be a 1EqUnk (p.25) or provide enough information for you to reduce more complex equations to 1EqUnks.

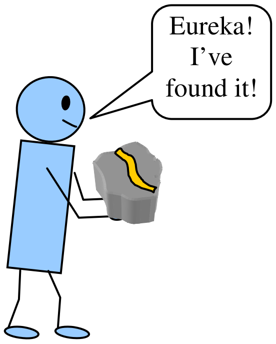
Extracting Gold

Imagine a muddy stream (word problem) with a dense jumble of rocks (words) containing traces of gold (1EqUnks).

Some gold is on the surface of the rocks and easily extracted.

Other gold is embedded in the rocks and requires special extraction tools.

In each type of equation on this page, the gold 1EqUnk is shaded.



1EqUnk

$x + 2 = 6$

One Equation with One Unknown contains loose gold, which requires no special tools to extract.

$$\begin{array}{lcl} x+2 & = & 6\\ \underline{-2} & & \underline{-2}\\ x & = & 4 \end{array}$$

2EqUnk

$x + y = 3$

$x - y = 1$

Two Equations with Two Unknowns contain loose gold, which requires no special tools to extract.

$$\begin{array}{lcl} x+y & = & 3\\ \underline{x-y} & = & \underline{1}\\ \underline{2x} & = & \underline{4}\\ 2 & & 2\\ x & = & 2 \end{array}$$

$$\begin{array}{lcl} x+y & = & 3\\ 2+y & = & 3\\ \underline{-2} & & \underline{-2}\\ y & = & 1 \end{array}$$

1Eq3Unk

$A = B + C$

One Equation with Three Unknowns has embedded gold that requires one of the following toolkits to extract.

Toolkit 1

2 values

$B = 6, C = 4$

$$\begin{array}{lcl} A & = & B + C\\ A & = & 6 + 4\\ A & = & 10 \end{array}$$

Toolkit 2

1 value

1 substitution

$A = 10$

$B = C + 2$

$$\begin{array}{lcl} A & = & B + C\\ 10 & = & C+2 + C\\ 10 & = & 2C + 2\\ \underline{-2} & & \underline{-2}\\ \frac{8}{2} & = & \frac{2C}{2}\\ 4 & = & C \end{array}$$

$$\begin{array}{lcl} A & = & B + C\\ 10 & = & B + 4\\ \underline{-4} & & \underline{-4}\\ 6 & = & B \end{array}$$

2Eq6Unk

$Q_1 = R_1K_1 \quad Q_2 = R_2K_2$

Two Equations with Six Unknowns have deeply embedded gold that requires one of the following toolkits to extract.

Toolkit 1

4 values

$R_1=2,\ K_1=6$

$R_2=3,\ K_2=4$

$$\begin{array}{lcl} Q_1 & = & R_1K_1\\ Q_1 & = & 2 \bullet 6\\ Q_1 & = & 12 \end{array}$$

$$\begin{array}{lcl} Q_2 & = & R_2K_2\\ Q_2 & = & 3 \bullet 4\\ Q_2 & = & 12 \end{array}$$

Toolkit 2

3 values

1 equality

$R_1=2, R_2=3, K_2=4$

$Q_1=Q_2$

$$\begin{array}{lcl} Q_1 & = & Q_2\\ R_1K_1 & = & R_2K_2\\ 2K_1 & = & 3 \bullet 4\\ \frac{2K_1}{2} & = & \frac{12}{2}\\ K_1 & = & 6 \end{array}$$

Toolkit 3

2 values

1 equality

1 substitution

$R_1=2,\ R_2=3$

$Q_1=Q_2$

$K_1= K_2+2$

$$\begin{array}{lcl} Q_1 & = & Q_2\\ R_1K_1 & = & R_2K_2\\ 2K_1 & = & 3K_2\\ 2(K_2+2) & = & 3K_2\\ 2K_2+4 & = & 3K_2\\ \underline{-2K_2} & & \underline{-2K_2}\\ 4 & = & K_2 \end{array}$$

$$\begin{array}{lcl} K_1 & = & K_2+2\\ K_1 & = & 4 + 2\\ K_1 & = & 6 \end{array}$$

FYI

1 equality + 2 substitutions

or

2 equalities + 1 substitution

may produce quadratic equations.

Unit Analysis

Unit Analysis can help you decide how to set up an equation to get the desired result. It ensures that your final answer will have the appropriate units before you spend time calculating.

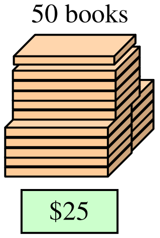
Consider the following, almost identical problems. You probably know that division is involved, but the dilemma is: Which way to divide? Unit analysis makes it much easier to decide.

If 50 books cost $25,
how much does one book cost?

Unit Analysis: cost/book

Divide: $25 cost / 50 books

Solution: $0.50 cost / 1 book



If 50 books cost $25,
how many can you buy for $1?

Unit Analysis: books/cost

Divide: 50 books / $25 cost

Solution: 2 books / $1 cost

Tip
In general, the item you are seeking goes on top (numerator)
and the per-unit item goes on the bottom (denominator).

Proportional Ratios

In the preceding problems, dividing with the given numbers (50 books and $25) produced correct answers because both problems asked for a quantity for *one*; i.e., cost for *one* book, books for *one* dollar. Since the result of a division is a *one* in the denominator, straight division worked.

When a problem asks for more than *one* in the denominator, use Proportional Ratios.

If 50 books cost $25, how much do 4 books cost?	**If 50 books cost $25, how many can you buy for $4?**
Let C = Cost	Let B = Books
$\frac{\$25}{50\text{ books}} = \frac{\$C\text{ cost}}{4\text{ books}}$	$\frac{50\text{ books}}{\$25} = \frac{B\text{ books}}{\$4}$
$\frac{\$25(4\ \cancel{\text{books}})}{50\ \cancel{\text{books}}} \xleftarrow{=} \$C$	$\frac{50\text{ books}(\cancel{\$}4)}{\cancel{\$}25} \xleftarrow{=} B\text{ books}$
$\boxed{\$2 = \$C}$	$\boxed{8\text{ books} = B\text{ books}}$

See Shoot-the-Chute (p.33).

Freeform Word Problems

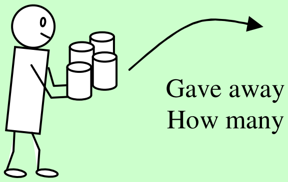
Instead of predefined formulas, some word problems require you to build equations directly from the text in the problem. This is when the English-to-Math Chart (p.57) helps the most. Freeform problems often involve 1EqUnk (p.25) or 2EqUnk (p.35) equations.

1EqUnk Problems

Joe bought 4 cans of nuts and gave half away as gifts. How many does he have left?

Identify: Freeform 1EqUnk problem

Draw:



Gave away half.
How many left?

$$\begin{array}{lcccccl} \text{Equate:} & \underline{\text{Cans left}} & = & \text{cans bought} & - & \tfrac{1}{2}\text{ cans bought.} & \\ \text{Assign:} & C & = & 4 & - & \tfrac{1}{2}(4) & \text{(all units in cans)} \\ \text{Solve:} & C & = & 4 & - & 2 & \\ & C & = & & \boxed{2\text{ cans}} & & \end{array}$$

Your turn: Meg picked 6 plums, gave 2 away, and picked 4 more. How many does she have now?

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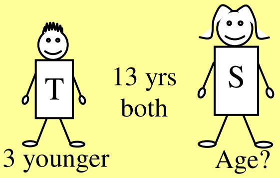
2EqUnk Problems

Tom is 3 years younger than Sue. Together they are 13 years old. How old is each?

Identify: Freeform 2EqUnk problem

Draw:



$$\begin{array}{lccccc} \text{Equate:} & \underline{\text{Tom}}_{\text{yrs}} & = & \underline{\text{Sue}}_{\text{yrs}} & - & 3_{\text{yrs}} \\ \text{Assign:} & T & = & S & - & 3 \\ \text{Solve:} & T & = & 8 & - & 3 \\ & T & = & & \boxed{5\text{ yrs}} & \end{array}$$

$$\begin{array}{ccccccl} \underline{\text{Tom}}_{\text{yrs}} & + & \underline{\text{Sue}}_{\text{yrs}} & = & 13_{\text{yrs}} & & \text{(yrs = years)} \\ T & + & S & = & 13 & & \text{(all units in yrs)} \\ S-3 & + & S & = & 13 & & \\ 2S & - & 3 & = & 13 & & \\ & & \underline{+3} & & \underline{+3} & & \\ \frac{2S}{2} & & & = & \frac{16}{2} & & \\ S & & & = & \boxed{8\text{ yrs}} & & \end{array}$$

Check

T + S = 13

5 + 8 = 13 √

Your turn: Bob has 2 more pens than Jan. Together they have 10. How many pens does each have?

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S

Geometric Word Problems

Perimeter Problems

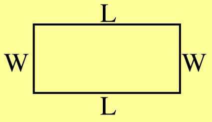
Perimeter [pur-IH-meh-tur] is a measure of the distance around an object.
Peri is Greek for "around." *Meter* is Greek for "measure."

Rectangle

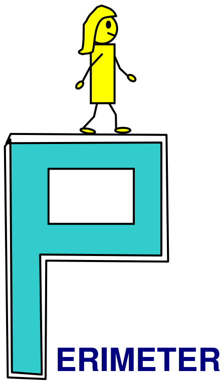
The perimeter of a rectangle is twice its length plus twice its width.

$P_R = 2L + 2W$

P_R = Perimeter of rectangle
L = Length (long side)
W = Width (short side)



Alternate Equation: Factoring out the 2 yields: $P_R = 2(L + W)$



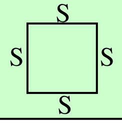
BrainAid
Imagine walking a path around the letter P.

Square

The perimeter of a square is four times the length of one side.

$P_S = 4S$

P_S = Perimeter of square
S = Length of one side



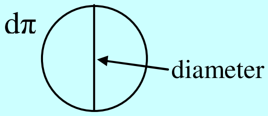
BrainAid
Imagine the letter 'C' circling the circumference.

Circle

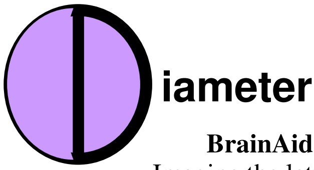
The perimeter, aka circumference [sur-CUM-frenss], of a circle is its Diameter times Pi [pii].

$C = d\pi$

C = Circumference
d = diameter (a line through the center)
π = pi = ~3.14 or ~22/7



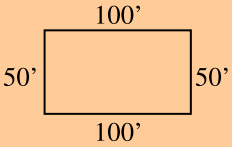
* The word pi is Greek for periphery and came from measuring circles.



BrainAid
Imagine the letter D creating the diameter.

How much fencing is needed to enclose a 100 ft by 50 ft field? (ft = foot or feet)

Identify: Rectangle Perimeter problem
Draw:



Equate: $\text{Perimeter}_{\text{rectangle}} = 2 \bullet \text{length} + 2 \bullet \text{width}$ (all units in feet)
Assign: $P_R = 2L + 2W$
Solve: $P_R = 2(100) + 2(50)$
$P_R = \boxed{300 \text{ ft}}$

Your turn: What is the distance around a village square that's 30m on each side? (m = meters)

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Area Problems

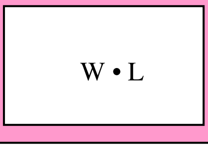
Area [AIR-ee-uh] is a measure of the space on the surface of an object.
Area is Latin for “level ground” or “open space.”

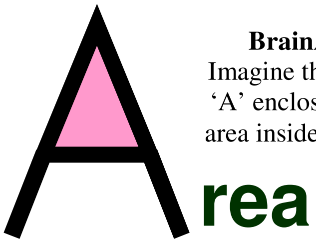
Rectangle

The area of a rectangle is its width times its length.

$A_R = WL$

A_R = Area of rectangle
W = Width (short side)
L = Length (long side)





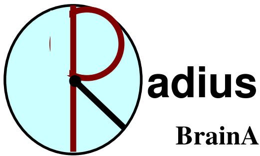
BrainAid
Imagine the letter
‘A’ enclosing the
area inside its top.

Square

The area of a square is the length of one side squared.

$A_S = S^2$

A_S = Area of square
S = Length of one side



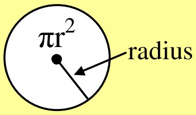
BrainAid
Imagine the slanted
leg of the letter ‘R’
as the radius.

Circle

The area of a circle is pi times its radius squared.

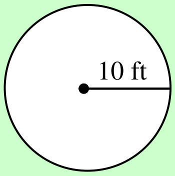
$A_C = \pi r^2$

A_C = Area of circle
r = radius (a line from center to edge = ½ diameter)
π = pi = ~3.14 or ~22/7

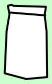


How many square feet is a circular lawn whose radius is 10 ft? (ft = foot or feet, ft^2 = square ft)

Identify: Circle Area problem
Draw:
Equate: $\text{Area}_{circle} = \underline{pi} \cdot \underline{radius}^2$
Assign: $A_C = \pi r^2$
Solve: $A_C = 3.14(10\text{ ft})^2$
$A_C = 3.14(100\text{ ft}^2)$
$A_C = \boxed{314\text{ ft}^2}$



Lawn
314 ft^2



Seed
157 ft^2

How many bags of seed are needed for this lawn if one bag covers 157 ft^2?

Identify: Freeform Division problem
Draw:
Equate: $\text{Bags} = \text{Area}_{ft}^2 / \text{Coverage}_{ft}^2$
Assign: $B = A / C$
Solve: $B = 314 / 157$
$B = \boxed{2\text{ bags}}$

Your turn: How many square yards is a tarp that measures 50 yd x 30 yd? (yd = yard/s, yd^2 = square yd)

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Travel Word Problems

Distance/Rate/Time: DRT

The Distance traveled equals the Rate of travel times the Time traveled.

D = RT

D = Distance traveled

R = Rate of travel (average speed)

T = Time traveled

Travel Rate: R = D/T

Travel Time: T = D/R

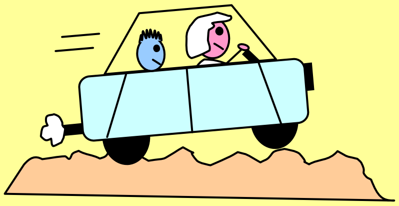
Alternate Equations

$$\frac{D}{T} = \frac{R\cancel{T}}{\cancel{T}}$$

$$\frac{D}{T} = R$$

$$\frac{D}{R} = \frac{\cancel{R}T}{\cancel{R}}$$

$$\frac{D}{R} = T$$



BrainAid

Pronounce DRT as “dirt.”

Imagine a car traveling on a dirt road.

What distance is traveled by a biker averaging 10 mph for 2 hours? (mph = miles per hour)

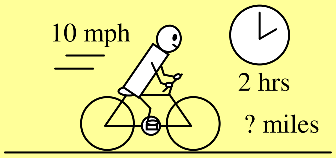
Identify: Travel Distance problem

Draw:

Equate: $\underline{\text{Distance}}_{\text{mi}} = \underline{\text{Rate}}_{\text{mph}} \bullet \underline{\text{Time}}_{\text{hr}}$ (mi=mile/s, hr=hour/s)

Assign: D = RT

Solve: $D = 10\text{ miles}/\cancel{\text{hour}} \bullet 2\ \cancel{\text{hours}} = \boxed{20\text{ miles}}$



A biker who rides 60 miles in 4 hours pedals how fast on average?

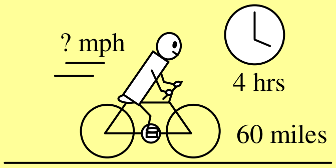
Identify: Travel Rate problem

Draw:

Equate: $\underline{\text{Rate}}_{\text{mph}} = \underline{\text{Distance}}_{\text{mi}} / \underline{\text{Time}}_{\text{hr}}$

Assign: R = D/T

Solve: $R = 60\text{ miles} / 4\text{ hours} = \boxed{15\text{ mph}}$



How long does a biker take to ride 12 miles at 3 mph?

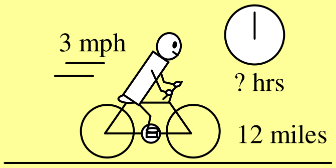
Identify: Travel Time problem

Draw:

Equate: $\underline{\text{Time}}_{\text{hr}} = \underline{\text{Distance}}_{\text{mi}} / \underline{\text{Rate}}_{\text{mph}}$

Assign: T = D/R

Solve: $T = 12\ \cancel{\text{miles}} / 3\ \cancel{\text{miles}}/\text{hours} = \boxed{4\text{ hours}}$



Tip

Instead of memorizing the Alternate Equations, remember D=RT, and isolate the unknown variable as needed with Shoot-the-Chute (p.33).

$$\frac{D}{T} \rightleftarrows R$$

$$\frac{D}{R} \rightleftarrows T$$

R and T always go together.

Your turn: How far does a biker ride when averaging 15 mph for 5 hours?

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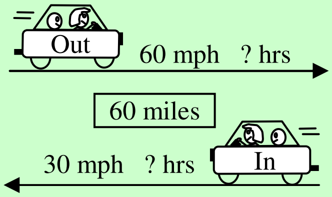
Double DRT: Round Trip Average Rate

The average Rate of travel for a round trip (or several shorter trips) is the *total* Distance divided by the *total* Time. $\mathbf{R_{avg} = D_{total} / T_{total}}$

What is the average rate of a car that travels 60 miles outbound @ 60 mph, then returns 60 miles inbound @ 30 mph? (mi = mile/s; mph = miles per hour, hr = hour/s)



INCORRECT

Identify: Average = Sum of Items / Total Items

Draw:

Equate: $\underline{\text{Rate}}_{\text{avg}} = (\underline{\text{Rate}}_{\text{out}} + \underline{\text{Rate}}_{\text{in}}) / 2$

Assign: $R_A = (R_O + R_I) / 2$

Solve: $R_A = (60\text{mph} + 30\text{mph}) / 2 = 90\text{ mph} / 2 = \cancel{\boxed{45\text{ mph}}}$

CORRECT

Identify: Travel Time problem (two trips)

Draw:

Equate: $\underline{\text{Time}}_{\text{out}} = \underline{\text{Distance}}_{\text{out}} / \underline{\text{Rate}}_{\text{out}}$ $\quad \underline{\text{Time}}_{\text{in}} = \underline{\text{Distance}}_{\text{in}} / \underline{\text{Rate}}_{\text{in}}$

Assign: $T_O = D_O / R_O$ $\quad T_I = D_I / R_I$

Solve: $T_O = 60\ \cancel{\text{mi}} / 60\cancel{\text{m}}\text{ph} = \boxed{1\text{ hr}}$ $\quad T_I = 60\ \cancel{\text{mi}} / 30\cancel{\text{m}}\text{ph} = \boxed{2\text{ hr}}$

Identify: Round Trip Average Rate

Draw:

Equate: $\text{Rate}_{\text{avg}} = \text{Distance}_{\text{total}} / \text{Time}_{\text{total}}$

$\text{Rate}_{\text{avg}} = (\underline{\text{Distance}}_{\text{out}} + \underline{\text{Distance}}_{\text{in}}) / (\underline{\text{Time}}_{\text{out}} + \underline{\text{Time}}_{\text{in}})$

Assign: $R_A = (D_0 + D_I) / (T_0 + T_I)$

Solve: $R_A = (60\text{ mi} + 60\text{ mi}) / (1\text{ hr} + 2\text{ hr}) = 120\text{ mi} / 3\text{ hr} = \boxed{40\text{ mph}}$

Trap!

45 mph, the midpoint between 60 mph and 30 mph, would be correct if the car traveled the same amount of time in both directions. But the car necessarily took longer to cover the inbound 60 miles at the slower 30 mph, which pulled the average down to 40 mph.

Your turn: What is the average rate of a car than travels 30 miles outbound @ 30 mph, then returns 30 miles inbound @ 10 mph?

I

D

E

A

S

I

D

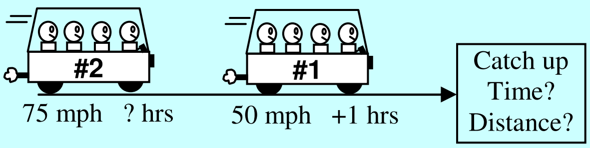
E

A

S

Double DRT: Catch Up

Bus#1 leaves the depot and averages 50 mph. Bus#2 leaves 1 hour later averaging 75 mph. How long will it take Bus#2 to catch up to Bus#1? At what distance from the depot?



Identify: Travel Catch Up problem (Distance equal)

Draw:

Equate: $\underline{\text{Distance}_1} = \underline{\text{Distance}_2}$ $\underline{\text{Time}_1} = \underline{\text{Time}_2} + 1\text{ hr}$ $\underline{\text{Rate}_1} = 50\text{mph}$ $\underline{\text{Rate}_2} = 75\text{mph}$

Assign: $D_1 = D_2$ $T_1 = T_2 + 1\text{ hr}$ $R_1 = 50\text{mph}$ $R_2 = 75\text{mph}$

$R_1T_1 = R_2T_2$ (substitute)

$R_1(T_2+1) = R_2T_2$

Solve: $50\text{mph}(T_2+1\text{hr}) = 75\text{mph}(T_2)$

$50T_2\text{mi} + 50\text{mi} = 75T_2\text{mi}$

$\underline{-50T_2\text{mi}} = \underline{-50T_2\text{mi}}$

$\frac{50\,\cancel{\text{mi}}}{25\cancel{\text{mph}}} = \frac{25T_2\,\cancel{\text{mi}}}{25\cancel{\text{mph}}}$

$\boxed{2\text{ hr}} = T_2$

$D_2 = R_2T_2$

$D_2 = 75\text{m}\cancel{\text{ph}} \bullet 2\cancel{\text{hr}}$

$D_2 = \boxed{150\text{ mi}}$

Check

$T_1 = T_2 + 1\text{hr}$

$T_1 = 2\text{hr} + 1\text{hr}$

$T_1 = 3\text{ hr}$

$D_1 = R_1T_1$

$D_1 = 50\text{m}\cancel{\text{ph}} \bullet 3\cancel{\text{hr}}$

$D_1 = 150\text{ mi}$ √

Solution

Bus2 catches up to Bus1 in 2 hours 150 miles from the depot.

Your turn: Ann leaves school and walks 2 mph. Bob leaves 1 hour later and walks 4 mph. How long will it take him to catch up with Ann and at what distance from school?

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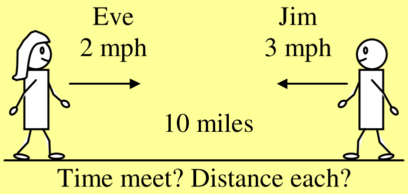
Double DRT: Meet in Between

Eve and Jim are 10 miles apart and start walking towards each other. Eve walks 2 mph. Jim walks 3 mph. How long does it take, and how far has each walked when they meet?



Identify: Travel Meet in Between (Time equal)

Draw:

Equate: $\underline{\text{Time}_{\text{Eve}}}=\underline{\text{Time}_{\text{Jim}}}=\text{Time}$ $\underline{\text{Distance}_{\text{Eve}}}+\underline{\text{Distance}_{\text{Jim}}}=10\text{ mi}$ $\underline{\text{Rate}_{\text{Eve}}}=2\text{mph}$ $\underline{\text{Rate}_{\text{Jim}}}=3\text{mph}$

Assign: $T_E = T_J = T$ $D_E + D_J = 10\text{ mi}$ $R_E = 2\text{mph}$ $R_J = 3\text{mph}$

Add equations to save steps

$$\begin{aligned} D_E &= R_E T \\ +\underline{D_J} &= \underline{R_J T} \\ (D_E+D_J) &= (R_E+R_J)(T) \end{aligned}$$

(substitute)

Why it works to add equations

$3 = 3$

$\underline{+\ 4 = 4}$

$3+4 = 3+4$

Solve: $10\text{ mi} = (2+3)\text{mph}(T)$

$$\frac{10\ \cancel{\text{mi}}}{5\cancel{\text{m}}\text{ph}} = \frac{5\cancel{\text{mph}}(T)}{5\cancel{\text{mph}}}$$

$$\boxed{2\text{ hr}} = T$$

$D_E = R_E T$

$D_E = 2\cancel{\text{mph}} \cdot 2\cancel{\text{hr}}$

$D_E = \boxed{4\text{ mi}}$

$D_J = R_J T$

$D_J = 3\cancel{\text{mph}} \cdot 2\cancel{\text{hr}}$

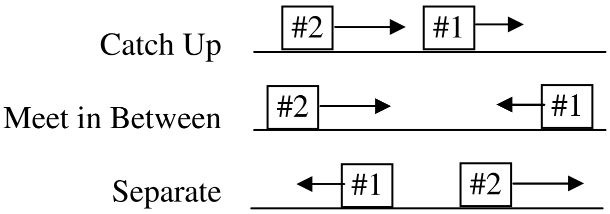
$D_J = \boxed{6\text{ mi}}$

Solution

They meet in 2 hours. Ann walked 4 miles; Bob walked 6 miles.

Double DRT Variations

Many variations of distance, rate, and time and the relationships between them are possible, e.g., travelers may leave at same or different times, total distance may be provided but not individual distances, rates may be given in terms of each other as in “twice as fast.” The variations seem endless but are always based on D = RT.



Tip

See Word Problem Analysis (p.58) for the minimum elements needed to solve 2Eq6Unk problems like Double DRTs.

Mileage: DMV

Distance traveled equals Mileage rate times fuel Volume.

D = MV

D = Distance in miles

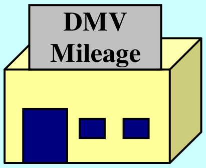
M = Mileage in miles per gallon (mpg)

V = Volume of fuel in gallons (gal)

Alternate Equations

M = D/V

V = D/M



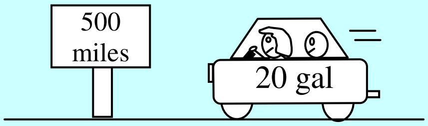
BrainAid

DMV stands for the Dept. of Motor Vehicles.

A car travels 500 miles on a 20-gallon tank of gas. What is its mpg?

Identify: Travel Mileage

Draw:



Equate: $\underline{\text{mpg}} = \underline{\text{Distance}}_{\text{mi}} / \underline{\text{Volume}}_{\text{gal}}$

Assign: M = D/V

Solve: $M = 500\text{ mi} / 20\text{ gal} = \boxed{25\text{ mpg}}$

Financial Word Problems

Besides financial items, these equations will work for almost any type of percent increase, decrease, or change problems. Tip: Review percents, fractions, and decimals in *Max Learning's Fraction Fun*.

Price Markup on Cost: NO+MO

Merchants mark up (raise) the price of a product so they can make a profit on each sale.

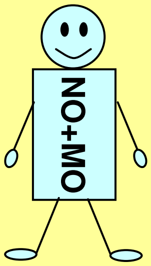
The New price equals the Old price plus the Markup% times the Old price.

$N = O + MO$

N = New price (aka Retail or List price)

O = Old price (aka Wholesale or Original price)

M = Markup Percent



BrainAid

Imagine a positive (+) merchant named NO+MO [noh-moh] who loves to Markup prices.

Alternate Equation: Factoring out the O yields: $N = O(1 + M)$.

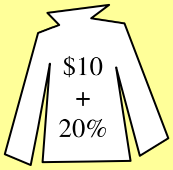
Explanation: $(1 + M)$ is the multiplier that yields the New price, e.g., if Markup = 20%, the New price is 120% of the Old.

The math: $(1 + 20\%) = (100\% + 20\%) = 120\%$

What is the price of a $10 coat after a 20% markup?

Identify: Price Markup problem

Draw:



Equate: New price = Old price + Markup% • Old price

Assign: $N = O + MO$

Solve: $N = \$10 + 20\%(\$10) = \$10 + \$2 = \boxed{\$12}$

Markup on Sell Price

Some merchants prefer to make markups based on the selling price, which results in a greater new price. They use this equation instead:

$N = O + MN$

Your turn: What is the price of a $20 coat after a 50% markup?

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Tax "Markup": NO+TO

Adding sales tax to an item is like marking it up by the tax percentage.

$N = O + TO$

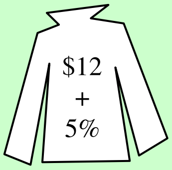
T = Tax percentage (replaces M)

Imagine a NO+MO has a cousin named NO+TO [noh-toh].

What is the price of a $12 coat with 5% sales tax?

Identify: Tax Markup problem

Draw:



Equate: New price = Old price + Tax% • Old price

Assign: $N = O + TO$

Solve: $N = \$12 + 5\%(\$12) = \$12 + \$0.60 = \boxed{\$12.60}$

Tip

To distinguish between the New markup price and the New price after tax, use different subscripts for N, e.g.,

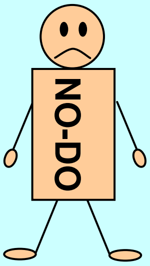
N_M = New markup price

N_T = New price after tax

Price Discount: NO–DO

Merchants discount (lower) the price of a product
to increase the number of items sold.

The New price equals the Old price minus the Discount% times the Old price.
N = O – DO
N = New price (aka Discounted or Sale price)
O = Old price (aka Retail, List, or Original price)
D = Discount Percent



BrainAid
Imagine a negative (–) merchant named NO-DO [noh-doh] who has to Discount prices.

Alternate Equation: Factoring out the O yields: N = O(1 – D).
Explanation: (1 – D) is the multiplier that yields the New price, e.g., if Discount = 30%, the New price is 70% of the Old.
The math: (1 – 30%) = (100% – 30%) = 70%

What is the price of a $10 T-shirt after a 25% discount?
Identify: Price Discount problem
Draw:

Equate: New price = Old price – Discount% • Old price
Assign: N = O – DO
Solve: N = $10 – 25%($10) = $10 – $2.50 = $7.50

Shortcut Solution
If 25% is deducted,
75% remains.
75%($10) = $7.50

Your turn: What is the price of a $20 T-shirt after a 50% discount?

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Your turn: What is the price of the discounted T-shirt (from above) with 10% sales tax?

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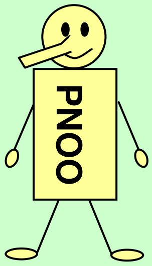
Percent-Change: PN-O/O

Merchants sometimes need to calculate the percent change between two prices to determine the markup or discount percent.

The Percent change equals the difference between the New price and the Old price divided by the Old price.

$$P = \frac{N - O}{O}$$

P = Percent change
N = New price (aka Current price)
O = Old price (aka Original or Base price)



BrainAid

Imagine PiNOcchiO, the puppet who became a boy, has the nickname PNOO [pi-noh]. He calculates the percent change in his nose size when he tells a lie.

$$Pi = \frac{N - O}{cchiO}$$

Markup-to-%Change Proof

$$\begin{aligned} N &= \cancel{O} + MO \\ -O & \quad \cancel{-O} \\ \frac{N - O}{O} &= \frac{M\cancel{O}}{\cancel{O}} \\ \boxed{\frac{N - O}{O}} &= M \end{aligned}$$

(% Markup)

Discount-to-%Change Proof

$$\begin{aligned} N &= \cancel{O} - DO \\ -O & \quad \cancel{-O} \\ \frac{N - O}{O} &= \frac{-D\cancel{O}}{\cancel{O}} \\ \boxed{\frac{N - O}{O}} &= -D \end{aligned}$$

(% Discount)

We could move the minus sign to the left side to make (–N + O) / O = D, but then you'd have to learn two equations. It's easier to use the same equation and remember that a negative result means a percent decrease or discount.

What is the percent markup on a $10 dress that now sells for $15?

Identify: Percent Change problem
Draw:
Equate: Percent-change = (New price – Old price) / Old price
Assign: P = (N – O) / O
Solve: P = ($15 – $10)/$10 = 5/10 = 50%

What is the percent discount on a $15 dress that's on sale for $10?

Identify: Percent Change problem
Draw:
Equate: Percent-change = (New price – Old price) / Old price
Assign: P = (N – O) / O
Solve: P = ($10 – $15)/$15 = -5/15 = -33%

The minus indicates a decrease. Because a discount is also a decrease, we'd say the percent discount is 33%, not -33%.

Percent Paradox

A *larger* Old price results in a *smaller* percent change.

Although the difference between the New and Old prices was $5 for both markup and discount, the percent changes were *not* the same, because they were based on different Old prices, first $10 (50% change) then $15 (-33% change).

Your turn: What is the percent discount on a $20 dress that's on sale for $15?

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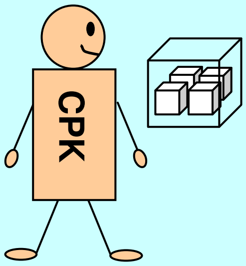
Cost: CPK

Merchants must often calculate the cost of selling or buying a quantity of identical items.

The Cost equals the Price per item times the Kwantity of items.

C = PK
C = Cost of all items
P = Price (aka Cost) for one item
K = Kwantity (made-up word) of items purchased



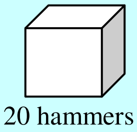
BrainAid
Imagine a merchant named CPK [see-pak] *see*ing a *pack*age of items he sold.

What is the total cost of 5 hammers sold for $10 each?

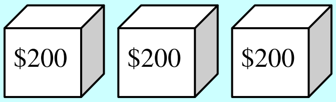
Identify: Cost problem
Draw:
Equate: Cost of hammers = price per hammer • kwantity of hammers
Assign: C = PK
Solve: $C = \$10/\cancel{\text{hammer}}\ (5\ \cancel{\text{hammers}}) = \boxed{\$50}$

If a box contained twenty $10 hammers, how much would 3 boxes cost?

Identify: Cost problem (1 of 2)
Draw:
Equate: Cost of box = price per hammer • kwantity of hammers per box
Assign: C = PK
Solve: $C = \$10/\cancel{\text{hammer}}\ (20\ \cancel{\text{hammers}}/\text{box}) = \$200/\text{box}$



Identify: Cost problem (2 of 2)
Draw:
Equate: Cost of 3 boxes = price per box • kwantity of boxes
Assign: C = PK
Solve: $C = \$200/\cancel{\text{box}}\ (3\ \cancel{\text{boxes}}) = \boxed{\$600}$



Your turn: What is the total cost of 4 toasters sold for $15 each?

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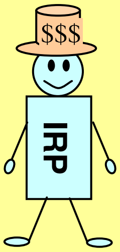
Also see **Unit Analysis** and **Proportional Ratios** (p.60) for alternative approaches to setting up and solving Cost and other problems.

Interest Earned: IRP

Interest earned on an investment equals the Rate (percent) of annual interest times the Principal invested.

I = RP

I = Interest earned ($)
R = Rate of annual interest (percent)
P = Principal invested ($)



BrainAid
Imagine Mr. IRP who proudly exclaims: "I ReaP interest from my investments."

How much interest does Ron earn in one year on a $1000 investment at 6%?

Identify: Interest Earned problem
Draw:

$1000 **6%** $Interest earned?

Equate: Interest earned = interest rate • principal
Assign: I = 6% • $1000
Solve: I = 6% • $1000 = 6/100 • $1000 = $60

The traditional equation is I=PR, but I=RP fits the Q=PK pattern (p.58).

I = RPT

T = Time period (in years or fraction of a year)

The traditional equation is I=PRT.

I=RP is derived from I=RPT where T=1. But any period can be used, e.g., T=2 equals 2 years. T=1/12 equals one month.
BrainAid: Mr. IRPT exclaims: "I reap interest RePeaTedly" over several periods.

I = RPT computes *simple* interest, which is calculated only on the originally invested Principal each period.

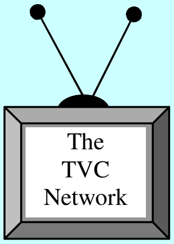
FYI: For *compound* interest, add the interest earned each period to the Principal, then compute the next period's interest on the new higher total. Compounding is a good thing for the investor since it increases the total interest earned.

Coins: TVC

The Total value of a group of the same-type coin is the Value of one coin times the number of those Coins.

T = VC

T = Total value
V = Value of one coin of that type
C = Coins of that type



BrainAid
Imagine TVC, the Total Value Channel, having a special sale on valuable coins.

What is the total value of Ned's 30 nickels?

T = $.05/n$\cancel{ick}$el • 30 n$\cancel{ick}$els = $1.50

Coin problems usually combine the total values of several coin equations.

- The variable 'C' changes for each coin type: P=Pennies, N=Nickels, D=Dimes, Q=Quarters.
- Total $_{\text{All coins}}$ = $.01P + $.05N + $.10D + $.25Q

Peg has a total of 7 dimes and quarters worth $1. How many of each does she have?

Identify: Coin problem
Draw:

7 coins: Dimes? ○ Quarters? ○ **$1**

Equate: Dimes + Quarters = 7 Total value = $.10 • Dimes + $.25 • Quarters
Assign: D + Q = 7 $1.00 = $.10D + $.25Q
Solve: -Q = -Q $1.00 = $.10(7-Q) + $.25Q
D =7-Q $1.00 = $.70–$.10Q + $.25Q
(substitute) $1.00 = $.70 + $.15Q
D = 7-2 -$.70 -$.70
D = ⑤
$.30 = $.15Q
$.15 $.15
② = Q

Solution
Peg has 5 dimes and 2 quarters.

Check
$1.00 = $.10D + $.25Q
$1.00 =$.10(5)+$.25(2)
$1.00 = $.50 + $.50
$1.00 = $1.00 √

Work Word Problems

Work/Rate/Time: WRT

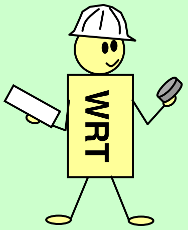
The Work completed equals the Rate of work times the Time worked.

W = RT

W = Work completed (aka job, task)
R = Rate of work
T = Time worked

Alternate Equations: R = W/T, T = W/R

WRT problems are very similar to DRT problems (p.64), except the “distance” traveled is the work completed.



BrainAid
Imagine WRT the Work Rate Timekeeper keeping track of how far workers have gone towards completing the work.

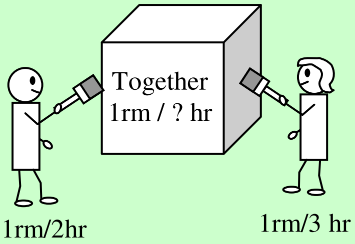
How many tasks can Rob complete if he performs 1 task in 2 hours and works for 10 hours?

$W = 1\text{ task}/2\ \cancel{\text{hours}} \cdot 10\ \cancel{\text{hours}} = 5\text{ tasks}$

Work problems usually combine the work rates of more than one worker.

Cal can paint 1 room in 2 hours. Zoe can paint 1 room in 3 hours.
How long does it take them to paint 1 room together?

rm = room/s
hr = hour/s



Identify: Work problem
Draw:
Equate: $\underline{Work_{Both}} = (\underline{Rate_{Cal}} + \underline{Rate_{Zoe}})\ \underline{Time}$

$$\begin{array}{lrcl}
\text{Assign:} & W & = & (R_C + R_Z)\ T \\
\text{Solve:} & 1rm & = & (1rm/2hr + 1rm/3hr)\ T \\
 & 6[1rm & = & (1rm/2hr + 1rm/3hr)\ T\ ] \longleftarrow \text{Clear Denominators (p.32.)} \\
 & 6rm & = & (3rm/hr + 2rm/hr)\ T \\
 & \dfrac{6\cancel{rm}}{5\cancel{rm}/hr} & = & \dfrac{(5\cancel{rm/hr})}{5\cancel{rm/hr}}\ T \\
 & \boxed{1\ 1/5\ hr} & = & T
\end{array}$$

Your turn: Ona can assemble one bike in 1 hour. Mac can assemble one bike in 2 hours. How long does it take for them to assemble one bike together?

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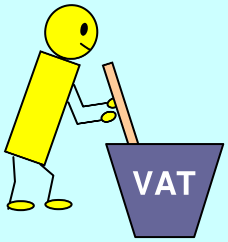
Mixture Word Problems

Volume/Amount/Total: VAT

The Volume of one component is an Amount (percent) of the Total mixture.



BrainAid
Imagine stirring a mixture in a large VAT.

V = AT
V = Volume of one component
A = Amount (percent)
T = Total volume of mixture

Alternate Equations: A = V/T, T = V/A

How many ounces (oz) of water are in a 100 oz beaker that's 25% water?
V = 25%(100 oz) = 25 oz

Typical mixture problems involve changes to the volume of components.

Ethanol is 40% of a 10-pint mixture. Tim adds 2 more pints of ethanol. What is its new percent?

Identify: Mixture problem
Draw:
Equate: Volume of ethanol = Amount % ethanol • Total volume
Volume ethanol+2 = Amount % ethanol+2 • Total volume+2

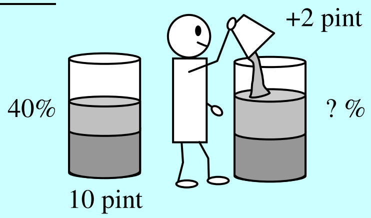
Assign: $V_E = A_E T$ $\quad V_{E2} = A_{E2} T_2$

Solve: $V_E = 40\%(10pt) = 4pt \longrightarrow (4+2pt) = A_{E2}(10+2pt)$

pt = pint/s

$$\frac{6\cancel{pt}}{12\cancel{pt}} = \frac{A_{E2}(12\cancel{pt})}{12\cancel{pt}}$$

$$\boxed{50\%} = A_{E2}$$



Your turn: Methyl is 10% of a 100-gallon mixture. Kai adds 20 more gallons of methyl. What is its new percent?

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Conversion Word Problems

A word problem may require you to convert one unit of measure into another.

Conversions: NCO

The number of New units equals the Conversion rate times the number of Old units.

BrainAid
Imagine NCO [nuu-coh] the magician making New COnversions.

N = CO
N = New units
C = Conversion rate
O = Old units

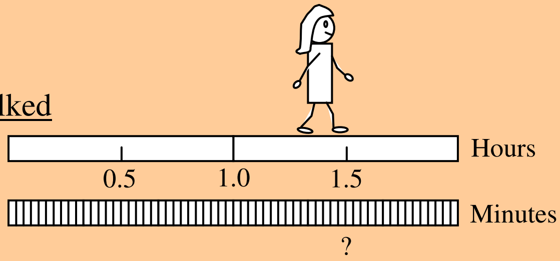
Alternate equations: C = N/O; O = N/C

The variables for N and O will change depending on the units being converted.

Rae walked for 1.5 hours. How many minutes did she walk?

Identify: Conversion problem

Draw:



Equate: Minutes walked = 60 minutes per hour • hours walked

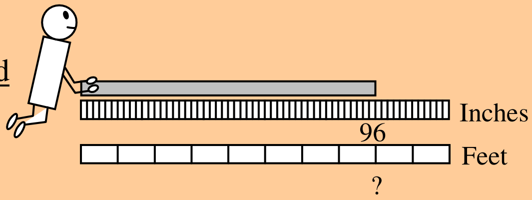
Assign: M = 60min/hr • H (min = minute/s; hr = hour/s)

Solve: $M = 60\text{min/}\cancel{\text{hr}} \bullet 1.5\cancel{\text{hr}} = \boxed{90\text{ min}}$

Nat bought a 96-inch piece of wood. How many feet is it?

Identify: Conversion problem

Draw:



Equate: Feet of wood = 1 foot per 12 inches • inches of wood

Assign: F = 1ft/12in • I (ft = foot/feet; in = inch/es)

Solve: $F = 1\text{ft/}12\cancel{\text{in}} \bullet 96\cancel{\text{in}} = \boxed{8\text{ ft}}$

Your turn: Eve has a 10-foot tree in her yard. How many inches tall is it?

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Conversion by Replacement/Ratio

Most conversions are usually part of a more complicated word problem and don't always merit the full IDEAS treatment. Below are two alternate conversion methods.

Conversion by Replacement

Replace the old unit with its equivalent in the new unit and multiply.

3 hours = ? minutes

Process: Replace "hours" with "60 minutes" and multiply.

Solution: 3 hours = 3(60 minutes) = $\boxed{180\text{ minutes}}$

24 inches = ? feet

Process: Replace "inches" with "1/12 foot" and multiply.

Solution: 24 inches = 24(1/12 foot) = $\boxed{2\text{ feet}}$

Why Replacement Works:

It's based on N=CO being reversed to OC=N.

Old units	•	Conversion rate	=	New units
$3\ \cancel{\text{hours}}$	•	$60\text{ minutes}/\cancel{\text{hour}}$	=	180 minutes
$24\ \cancel{\text{inches}}$	•	$1/12\text{ foot}/\cancel{\text{inch}}$	=	2 feet

Your turn: 120 min = ? hr

Tip: 1 min = 1/60 hr

Your turn: 2 yds = ? ft

Conversion by Ratio

Set the New/Old ratio to the Conversion-rate ratio.
Solve for the New unit. Tip: Use Shoot-the-Chute (p.33).

How many seconds (S) are in 10 minutes?

$$\frac{S}{10\text{ min}} = \frac{60\text{ sec}}{1\text{ min}}$$

$$\frac{S}{10\text{ min}} \nearrow \frac{60\text{ sec}}{1\text{ min}}$$

$$S = \frac{(10\cancel{\text{min}})(60\text{sec})}{1\cancel{\text{min}}}$$

$$S = \boxed{600\text{ sec}}$$

Why Ratios Work:

They're based on N=CO being altered to N/O = C.

Inverse Ratios

In problems that place the unknown variable in the denominator of the ratio, make sure the units in the conversion-rate ratio match top to bottom. See Unit Analysis (p.60).

Your turn: How many feet (F) are in 5 yards?

Conversion Ladders

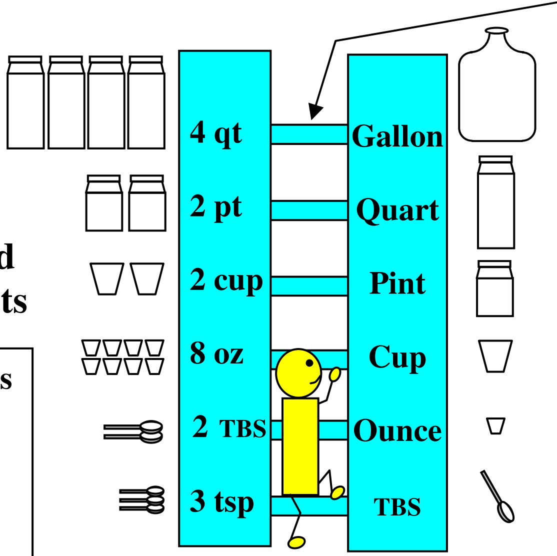
Ladders make multistep conversions easy!

Equivalents

Each ladder rung links
equivalent units
(e.g., 4 qt = 1 Gallon)
and yields
a Conversion rate
(e.g., C = 4 qt/Gallon).

U.S. Liquid Ladder

Old Units



New Units

tsp-teaspoon; TBS-tablespoon
oz-ounce; pt-pint; qt-quart

Multistep Conversions

- Start at the Old unit on the left side.
- Multiply up the left side until you reach the New unit's rung.

Water Weight

1 pt = 1 lb
1 qt = 2 lb
1 gal = 8 lb

pt-pint
lb-pound/s
gal-gallon

How many teaspoons in an ounce?

Procedure: Start at 3 tsp. Climb/multiply to 2 TBS.

Solution: $3 \cdot 2 = \boxed{6}$

Why it works: $3\,\text{tsp}/\cancel{\text{TBS}} \cdot 2\,\cancel{\text{TBS}}/\text{oz} = 6\,\text{tsp/oz}$

Your turn: How many ounces in a quart?

U.S. Linear Ladder

1760 yd	Mile
3 ft	Yard
12 in	Foot

in-inch/es
ft-foot/feet; yd-yard/s

How many feet in a mile?

$3 \cdot 1760 = \boxed{5280}$

Your turn: How many inches in a yard?

Tip

Larger dictionaries often contain conversion charts that compare U.S. to Metric units of measure.

Look under "measure" or in the appendix.

Trap!

Some conversion charts list conversion equations like:

miles × 1.6 = kilometers

This does *not* mean that 1.6 miles = 1 kilometer.

It means OC = N where
O = Old units = miles
C = Conversion rate = 1.6 km/mi
N = New units = kilometers

In fact,
1 mile = 1.6 kilometers.

Time Ladder

365 day	Year
24 hr	Day
60 min	Hour
60 sec	Minute

sec-second/s
min-minute/s; hr-hour/s

How many seconds in a day?

$60 \cdot 60 \cdot 24 = \boxed{86400}$

Your turn: How many minutes in a day?

Answer Key

One Equation, One Unknown

Page 25: 1EqUnk Added Term
Top Row: $x = 1$, $x = 3$; Bottom Row: $x = 4$, $x = 9$

Page 26: 1EqUnk Subtracted Term
Top Row: $x = 7$, $x = 9$; Bottom Row: $x = 14$, $x = 21$

Page 27: 1EqUnk Multiplied Variable
Top Row: $x = 3$, $x = 2$; Bottom Row: $x = 5$, $x = 4$

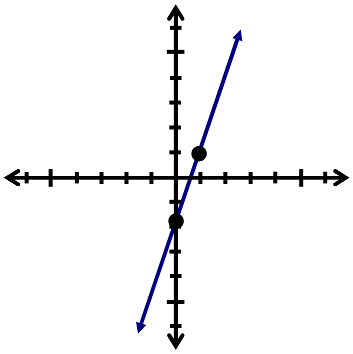
Page 28: 1EqUnk Divided Variable
Top Row: $x = 12$, $x = 3$; Bottom Row: $x = 8$, $x = 15$

Page 29: Multiple Operations: Clear As Mud
$x = 5$; $x = 3$

Page 30: Multiple Terms: Family Reunion
Top Row: $7x - 5$, $x^2 - 2x + 9$; Bottom Row: $-3x^2 + 5x + 1$, $-4x^2 - x + 3$

$y = 3x - 2$

$b = -2$
$m = 3/1$



Page 31: Separated Terms: Take Sides / Distributed Terms: Fair to All
Separated: $2x = 8$. Distributed: $-5x + 10 = 25$, $5x - 10 = 25$, $x - 2 = 25$

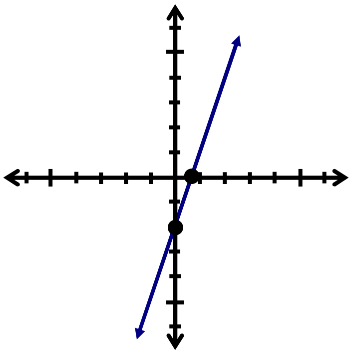
Page 32: Simplifying Coefficients: Clear Denominators / Reduce Coefficients
Clear: $5x + 5 = 2$, $x + 3 = 4$. Reduce: $3x + 1 = 4$, $x + 2 = -3$

Page 33: Clearing Equated Fractions: Shoot-the-Chute
$x = 5/3$; $x = 9/14$; $x = 8/15$

Page 34: Combining Fractions: Spotlighting
Left column: $7x/10 = 3$, $x/6 = 8$.
Right column: $5x/6 = 7$, $x/8 = 9$

$-3x+y = -2$

x	y
0	-2
2/3	0



Two Equations, Two Unknowns

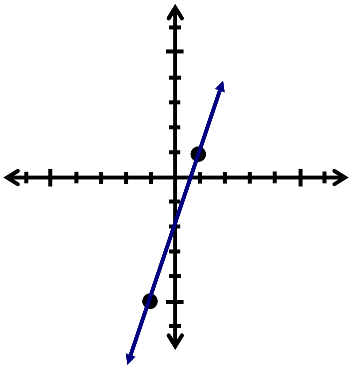
Page 38: Eliminate to Solve
Add: (3, 1). Subtract: (1, 2)

Page 39: Multiply Then Eliminate
Left: $10x=8$ or $5y=3$. Center: $3x=-13$ or $3y=14$. Right: $12x=25$ or $12y=7$

Page 40: 2EqUnk Substitution: Masquerade
(2, 4)

$-3x+y = -2$

x	y
-1	-5
1	1



Linear Equations

Page 42: Slope-Intercept Form
$y = 2x + 3$, $m = 2$, $b = 3$

Page 44: Calculating Slope: $\Delta y / \Delta x$
$m = 2$, $m = -2$, $m = 2$

Page 46: Plotting LinEqs
See plots on this page.

Quadratic Equations

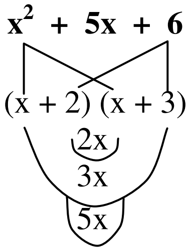
Page 49: QuadEq Traditional Techniques

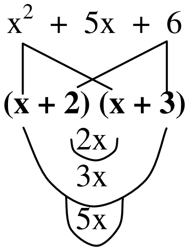
Left column: $x^2 + 3x$, $x(x + 3)$, $2x^2 + 6x$, $2x(x + 3)$

Center column: $x^2 + 5x + 6$, $(x + 2)(x + 3)$.

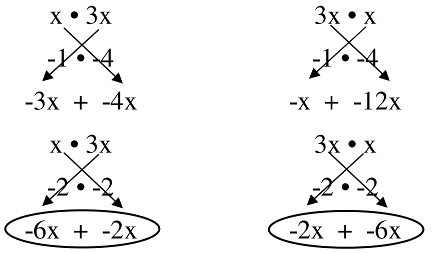
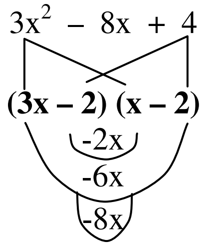
Right column: $x^2 – 9$, $(x + 3)(x – 3)$

Page 50: QuadEq Cat Techniques





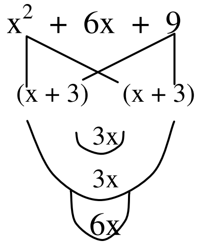
Page 51: QuadEq Cat Traps & Tips

List Ingredients	Prepare Food	Feed Cat
$\underline{3x^2} – 8x + \underline{4}$ x • 3x -1 • -4 3x • x -2 • -2		

Page 52: Zero-Product Principle

Left: x = 0 and/or x = -3. Center: x = 3 and/or x = -2. Right: x = ½ and/or x = 2.

Page 53: Solving QuadEqs by Cat Factoring

List Ingredients	Prepare Food	Feed Cat
$\underline{x^2} + 6x + \underline{9}$ x • x 1 • 9 3 • 3		

Apply Zero-Product Principle	Check Solution/s
$(x + 3)(x + 3) = 0$ $x + 3 = 0$ $\boxed{x = -3}$	$x^2 + 6x + 9 = 0$ $(-3)^2 + 6(-3) + 9 = 0$ $9 - 18 + 9 = 0$ $-9 + 9 = 0$ $0 = 0$ √

Page 54: Solving QuadEqs with Quadratic Formula

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)} = \frac{-3 \pm \sqrt{17}}{2}$$

Word Problems

Page 61: Freeform Word Problems

1EqUnk: $Meg_{plums} = 6 - 2 + 4 = 8$.

2EqUnk: $Bob_{pens} = 2 + Jan_{pens}$; $Bob_{pens} + Jan_{pens} = 10$. $Bob_{pens} = 6$; $Jan_{pens} = 4$.

Page 62: Geometric Word Problems

Perimeter (p.62): $Perimeter_{square} = 4 \bullet Side = 4(30) = 120$ m

Area (p.63): $Area_{rect} = Length \bullet Width = 50 \bullet 30 = 150\ yd^2$

Page 64: Travel Problems

DRT (p.64): Distance = 15 miles/hour • 5 hours = 75 miles

Double DRT: Round Trip Average Rate (p.65): The average rate for the round trip was 15 mph.
$T_{out} = D_O/R_O = 30$ mi / 30 mph = 1 hr; $T_{in} = D_I/R_I = 30$ mi / 10 mph = 3 hr
$R_{avg} = (D_O + D_I) / (T_O + T_I) = (30$ mi + 30 mi) / (1 hr + 3 hr) = 60 mi / 4 hr = 15 mph

Double DRT: Catch up (p.66): Bob catches up to Ann in 1 hour 4 miles from school.
$D_{Ann}=D_{Bob}$; $T_A=T_B+1$; $R_A(T_B+1)=R_BT_B$; $2mph(T_B+1)=4mph(T_B)$; $T_B = 1hr$; $D_B= 4mph \bullet 1hr = 4mi$

Page 68: Financial Word Problems

Price Markup on Cost (p.68): N = $20 + 50%($20) = $20 + $10 = $30

Price Discount (p.69): N = $20 – 50%($20) = $20 – $10 = $10; N_{tax} = $10 + 10%($10) = $10 + $1 = $11

Percent-Change (p.70): P = ($15 – $20) / $20 = -5/20 = -25/100 = -25%

Cost (p.71): C = $15/toaster • 4 toasters = $60

Page 73: Work Word Problems

WRT: $W_{Both}=(R_{Ona}+R_{Mac})T_{Both}$; 1bk=(1bk/hr+1bk/2hr)T; T = 2/3 hr (bk=bike)

Page 74: Mixture Word Problems

VAT: V_M=10%(100gal)=10gal. (10+20gal)=A_{M20}(100+20gal). A_{M20} = 25%

Page 75: Conversion Word Problems

NCO (p.75): Inches = 12in/ft • 10ft = 120 inches

By Replacement (76): 120min(1/60hr) = 2hr. 2yd(3ft) = 6ft

By Ratio (76): F/5yd = 3ft/1yd; F = 15ft

Liquid Ladder (p.77): 8 • 2 • 2 = 32 oz/qt

Linear Ladder (p.77): 12 • 3 = 36 in/yd

Time Ladder (p.77): 60 • 24 = 1400 min/day

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