Max Learning's Mental Math

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Welcome!

Hi, my name is Max Learning, and I'll be your teacher and guide.

My goal is to make math seem "real" to you, so you'll gain confidence and *look forward* to your next math challenge.

The fact that you're reading this book means you're eager to succeed and are willing to explore new ways to do so. *So let's get started!*

Why Is Math A Struggle?	How This Book Can Help
Symbols Math uses symbols, *lots* of them. It's as difficult to learn as a foreign language.	**Mental Manipulatives** You'll learn to "see" three-dimensional objects behind each symbol.
Rules Math is based on rules, *lots* of them. It's hard not to confuse one for the other.	**BrainAids** You'll learn clever memory hints that make the rules easy and fun.
Trauma Getting an answer wrong in front of the class, losing at a flash-card competition, failing a test, being criticized by a teacher—all can lead to math trauma.	**RUFF** You'll learn to be in a Relaxed, Uncluttered, Focused, and Flowing state of mind, which increases confidence and eases past traumas.

What's Good About Math?

Certainty
Math problems have *right* answers. An essay you wrote for English class, or a project you made for Art class, might seem fabulous to you, but maybe not to your teachers. However, in math, when you get the right answer, no one can argue with it.

Quest
Math problems are puzzles. The quest to solve them can be exciting! Math can be more fun than any game you'll ever play. If math becomes fun, you'll look forward to, rather than run from, it.

Magic
Math is the *language of nature*. Nearly everything we see, hear, touch and do can be described with math. Every image, color, and sound on a computer is based on math. In today's movies, you can't always tell what's real and what's been generated by some mathematical formula. In short, math is amazing—there's magic in it!

Note to Readers

For teaching/learning purposes, I've kept the demonstration problems in this book relatively simple, which may tempt you to solve them using traditional methods.

But for maximum results, it's important to take the time to learn and use the illustrated techniques. This may slow you down at first, but will pay off in the end. After all, it doesn't matter how quickly you solve a problem, if your answer is wrong!

You're learning a new, I hope, more interesting way of doing math, a way that links math symbols to real objects, if only in your imagination. As with learning anything new, it's best not to rush; so relax, take your time, and enjoy the process.

As your mind begins to "see" tangible objects behind the numbers and symbols, your speed and accuracy will improve, and you'll be ready to tackle more complex problems.

For some of the **Your turn** activities, I paradoxically ask you to write down what you are *thinking* as you mentally solve a problem. Why? So you can compare your thought processes to the **Answer Key** in the back of the book.

Note to Teachers

It's popular in some math classes to teach numerical concepts using physical objects, like blocks, tiles, and other toy-like objects. Many students enjoy working with these "manipulatives" because they make math seem real, even fun.

However, you've undoubtedly discovered that success with manipulatives does not always translate into success with purely symbolic math. A key objective of this book is to teach students to visualize *mental manipulatives*, so that math symbols are seen as physical entities even when the "toys" are put away.

Because this is a *techniques* book rather than a *drill & practice* book, it contains relatively few practice problems. However, once learned, students should be able to apply the same techniques to the numerous practice problems in traditional math textbooks, or to problems you make up for them to solve.

Pronunciation Guide

Sometimes it may not be obvious how to pronounce terms you have not heard spoken. When you see a term followed by a pronunciation, refer to this guide as needed.

Vowels		
Long	**Short**	**Other**
aa = ate	a = act	ai=air, ar=are, aw=paw
ee = eel	e/eh = end	
ii = hi	i/ih = hid	
oh = no	aw = on	oo = book, or = for
		ow = how, oy = boy
yu = use	u/uh = up	uu = too, ur = fur

Consonants	
Hard	**Soft**
k = cat	s = ice
g = go	j = gem
s/ss = hiss	z = his
ch = chin	sh=shin; zh=vision
th = thin	thh = this
Accent on: UP-ur-KAASS	

Common Abbreviations

aka = also known as
e.g. = for example (think egzample)
i.e. = that is

BrainAids

It was a mouthful to say *mnemonic* (nee-MAWN-ik) *device,* so I coined the word *BrainAid* for memory hints that help you learn. Feel free to make up your own BrainAids. Most fall into one of the following "A" categories:

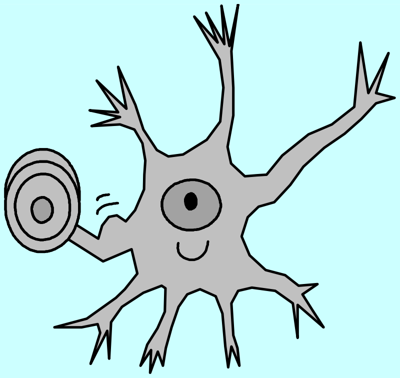
Analogy = Comparison

How to say it: uh-NOWL-uh-jee

What it is: A *comparison* of what you are trying to learn to what you already know.

Why it works: To learn, new brain fibers must grow and laboriously push their way through dense brain tissue, which can be very tiring. An analogy lets new information hitchhike along *existing* brain fibers, which is quicker and takes much less effort.

Analogy Example: Just as *physical* exercise builds new *muscle* fibers, *mental* exercise builds new *brain* fibers. Both take time, effort, and repetition.



Analogy: Building brain fibers.

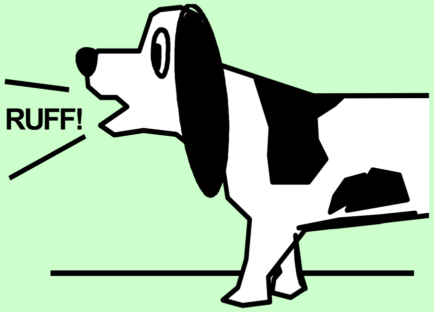
Acronym = Name

How to say it: AK-roh-nim

What it is: A *name* made from the first letters of several words. Hint: Think *nym* = *name*.

Why it works: The letters of an acronym act like small hooks on which you can hang large words. You memorize a little to remember a lot.

Acronym Example: To maximize your learning, be in a learning frame of mind: Relaxed, Uncluttered, Focused, and Flowing. In other words, be RUFF.



Acronym: RUFF

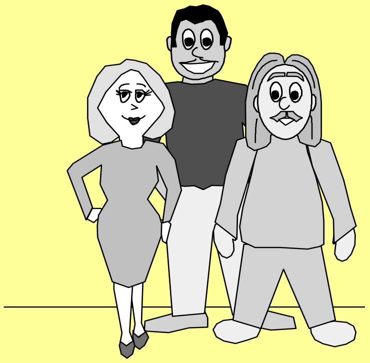
Acrostic = Story

How to say it: uh-KRAW-stik

What it is: A *story* made from the first letters of several words. Hint: Think *stic* = *st*ory.

Why it works: Sometimes a "story" is easier to remember than an acronym, especially if the acronym is hard to pronounce.

Acrostic Example: You create the acronym MTF to help you remember the names of three new acquaintances, Mary, Tom, and Fred. But MTF is hard to pronounce, so you convert it into the acrostic "My Three Friends."

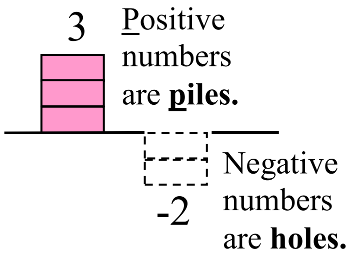


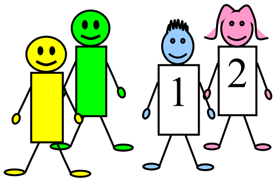
Acrostic: My Three Friends

Concepts

Mental Manipulatives

Traditional manipulatives are physical objects, like tiles or blocks, which you "manipulate" to mimic math operations. *Mental* manipulatives are items you visualize when you see a number or operation. They can turn lifeless symbols into reality—at least in your imagination. And unlike physical manipulatives, mental manipulatives are always with you to make math more engaging.





MathBots manipulate piles and holes or represent numbers.

Numbers

A number is a *symbol* for a quantity or value.

Numbers are symbols composed of individual *numerals* or *digits*.
Numbers don't exist in nature; only things do.
But things have quantity or value, which we represent with numbers.
BrainAid: Number = Sqv [skwiv]: Symbol for a quantity or value.

Natural Numbers

+

Natural Numbers are the Counting numbers: 1, 2, 3....
BrainAid: *Natural* numbers are used to count *natural* items, like sticks or stones.

- Positive Integers—Another name for the Natural Numbers.
- Cardinal Numbers—Another name for the Natural Numbers. (Think cardinal = counting.)
- Ordinal Numbers—Natural Numbers in ranked order: 1st, 2nd, 3rd… (Think ordinal = ordered.)

Whole Numbers

0+

Whole Numbers include Zero and the Counting numbers: 0, 1, 2, 3...
BrainAid: WhOle numbers include zerO.

Integers (IN-teh-jurz)*

Integers include the negatives of the Counting numbers, Zero, and the Counting numbers:

...-3, -2, -1, 0, 1, 2, 3...

BrainAid: Integers include negatives.

–0+

Whole Number and Integer are often used interchangeably to mean a number that is not a fraction, decimal, or percent; i.e., not part of a whole. However, Whole Numbers include only zero and the *Positive* Integers.

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* See Pronunciation Guide on page 4.

Place Values

The place of a digit in a number determines its value. Digits on the right have the least value; digits on the left have the most value. Observe below how the digits in 4025 are individually valued.

Place-Value Table

Million 1,000,000	hundred Thousand 100,000	ten Thousand 10,000	Thousand 1,000	hundred 100	ten 10	one 1
			4	0	2	5

4 Thousands = 4000
0 hundreds = 0
2 tens = 20
5 ones = + 5
4025

Zeros act as placeholders in the Place-Value Table, letting us write numbers of any size using only the digits 1-9. Before the invention of zero, all numbers larger than 9 required their own symbols, which made for a very cumbersome number system.

Operators

An operator is a *symbol* for a procedure or relationship.

Numbers by themselves do little unless we use operators to combine them in some way.
BrainAid: Operator = Spr [spur]: Symbol for a procedure or relationship.
You need to 'spur' sqvs (numbers) on to get them to work together.

Arithmetic Operators

Arithmetic operators specify procedures.
The following appear on a 4-function calculator.

+	Add
–	Subtract
×	Multiply
÷	Divide

Relational Operators

Relational operators specify relationships.
They include the following.

=	Equal
≠	Not equal to
>	Greater than
<	Less than
≥	Greater than or equal to
≤	Less than or equal to

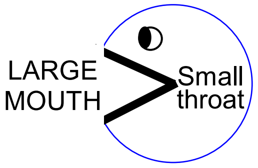
Computer Operators

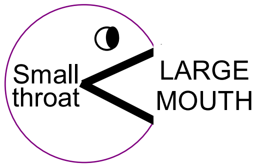
Many of the common operators do *not* appear on computer keyboards. Below are alternates, typically used in computer spreadsheet formulas.

*	Asterisk (aka star) for multiply
/	Slash for divide
^	Caret [KAIR-et] for exponentiation.
<>	Not equal to
>=	Greater than or equal to
<=	Less than or equal to

BrainAid

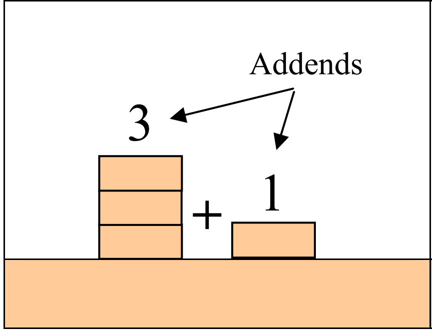
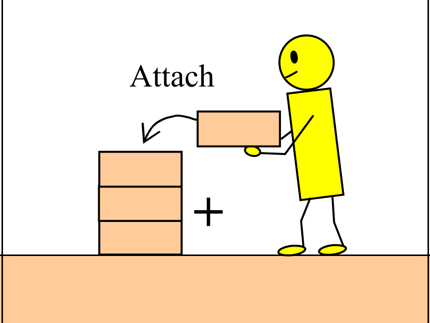
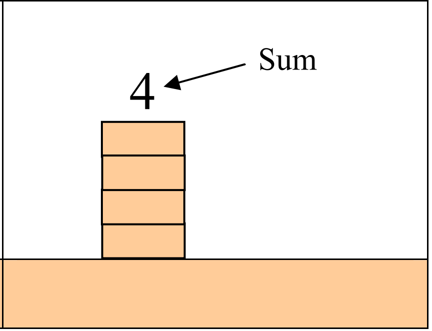
Be careful not to confuse the > and < symbols.
The *larger* number goes on the *larger* side.
Example: 7 > 6; 6 < 7



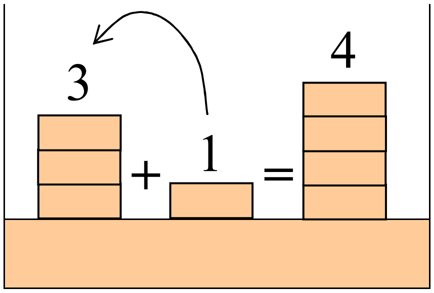


Addition Attaches

Larger Pile (3 + 1 = 4)

ADDENDS Each number to be added is called an *addend*. Think of something you *add* to the *end*.	OPERATOR The plus sign is an operator that says to 'attach.' *Plus* is Latin for *more.*	SUM The result of an addition is called the *sum*. Think of it as the SUMmary of the addends.
		

Now that the MathBot has shown us how to attach piles, we can redraw the addition this way.



Try it: Use piles and an arrow to sketch the following addition: 2 + 1 = 3

Your turn: Imagine attaching piles. "See" the piles. Fill in the blanks.

$\overset{\curvearrowleft}{3 + 2} =$ ___

$\overset{\curvearrowleft}{5 + 2} =$ ___

$\overset{\curvearrowleft}{6 + 2} =$ ___

If the smaller pile is on the left, attach it to the top of the larger pile on the right.

$\overset{\curvearrowright}{2 + 7} =$ ___

$\overset{\curvearrowright}{2 + 4} =$ ___

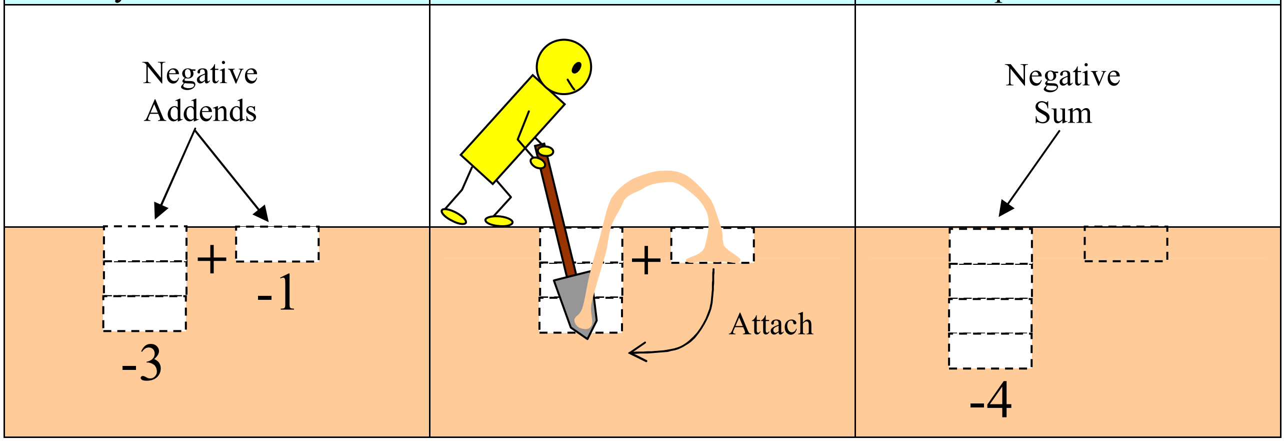
$\overset{\curvearrowright}{1 + 9} =$ ___

Larger Hole (-3 + -1 = -4)

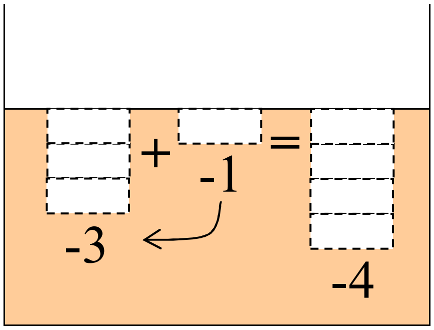
To ‘attach’ holes, you can’t carry one to the other…

…so dig the deep hole deeper and fill in the shallow one.

The shallow hole is gone. A deeper hole remains.



We can redraw this addition more compactly.



Try it: Use holes and an arrow to sketch the following addition: -2 + -1 = -3

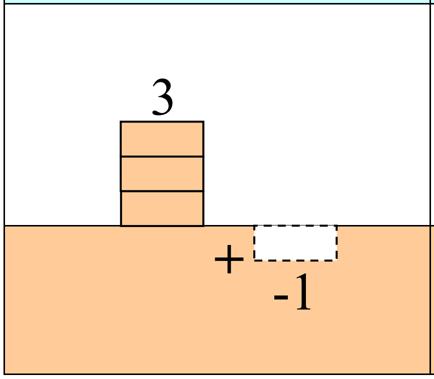
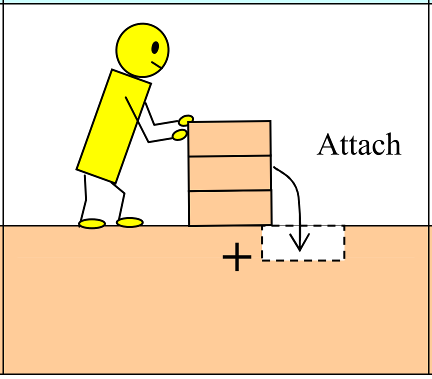
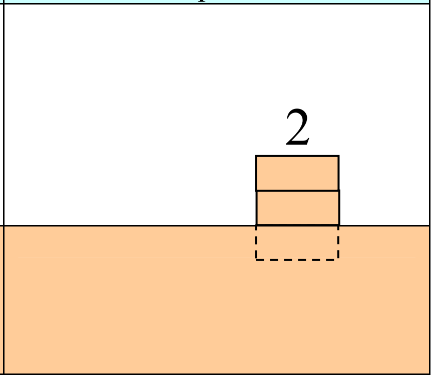
Your turn: Imagine attaching holes. Check your answers in the back of this book.

$\underleftarrow{-3 + -2}$ = ___ $\underleftarrow{-5 + -2}$ = ___ $\underleftarrow{-6 + -2}$ = ___

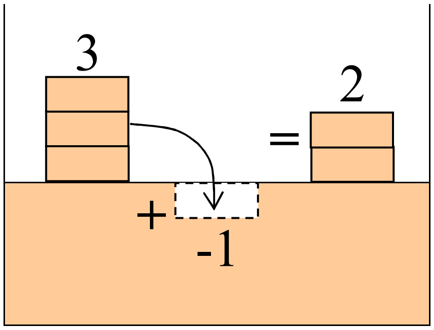
If the shallower hole is on the left, attach it to the bottom of the deeper hole on the right.

$\underrightarrow{-2 + -7}$ = ___ $\underrightarrow{-2 + -4}$ = ___ $\underrightarrow{-1 + -9}$ = ___

Smaller Pile (3 + -1 = 2)

To attach a tall pile and a shallow hole...	…push the pile into the hole.	The hole is filled and gone. A shorter pile remains.
		

Drawn more compactly, the addition would look like this.



Try it: Use piles, a hole, and an arrow to sketch the following addition: 2 + -1 = 1

Your turn: Imagine pushing a tall pile into a shallow hole. "See" yourself doing it.

$\overset{\curvearrowright}{3 + -2}$ = ___

$\overset{\curvearrowright}{5 + -2}$ = ___

$\overset{\curvearrowright}{6 + -2}$ = ___

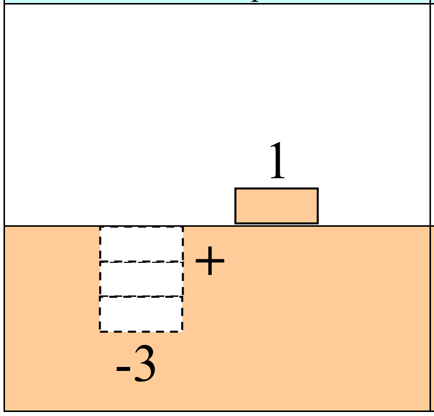
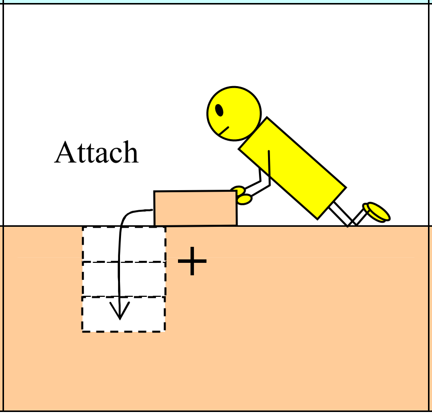
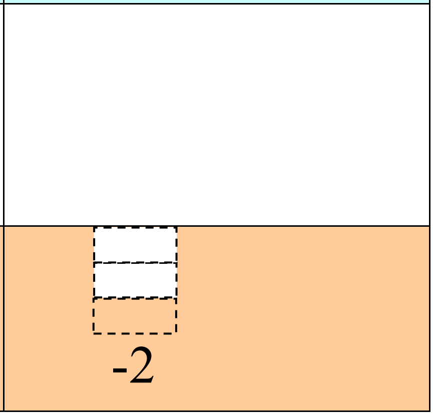
If the hole is on the left, push the pile from the right.

$\overset{\curvearrowleft}{-2 + 7}$ = ___

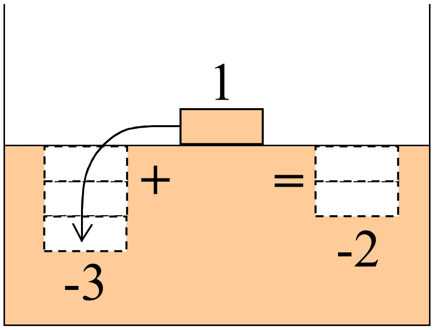
$\overset{\curvearrowleft}{-2 + 4}$ = ___

$\overset{\curvearrowleft}{-1 + 9}$ = ___

Smaller Hole (-3 + 1 = -2)

To attach a deep hole and a short pile…	…push the pile into the hole.	The pile is gone. A shallower hole remains.
		

Here’s the compact version.



Try it: Use a pile, holes, and an arrow to sketch the following addition: -2 + 1 = -1

Your turn: Visualize pushing a short pile into a deep hole.

$\overset{\curvearrowleft}{\mathbf{-3 + 2}} =$ ___ $\overset{\curvearrowleft}{\mathbf{-5 + 2}} =$ ___ $\overset{\curvearrowleft}{\mathbf{-6 + 2}} =$ ___

If the hole is on the right, push the pile from the left.

$\overset{\curvearrowright}{\mathbf{2 + -7}} =$ ___ $\overset{\curvearrowright}{\mathbf{2 + -4}} =$ ___ $\overset{\curvearrowright}{\mathbf{1 + -9}} =$ ___

Properties of Addition

Properties are the *rules* of the game.
These rules make the mental math tricks in this book possible.

Commutative Property of Addition: Changing Order

Property: Addends can be added in any *order*.

BrainAid: Commutative [pronounced cuh-MYU-tuh-tiv]* comes from *commute* which can mean to change an order, as in “to commute a prisoner’s sentence,” or to change the order of travel, as in “to commute from home to office in the morning, then from office to home in the evening.” Think of the “co” in commutative as meaning change order.

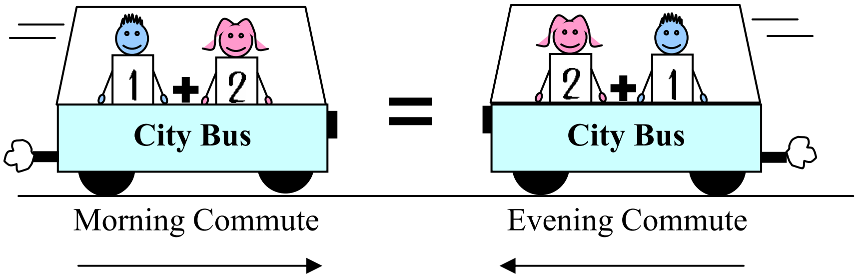
$1 + 2 = 3$
$2 + 1 = 3$

Changing the order does not change the sum.

With three addends:

$1 + 2 + 3 = 6$
$1 + 3 + 2 = 6$
$2 + 1 + 3 = 6$
$2 + 3 + 1 = 6$
$3 + 1 + 2 = 6$
$3 + 2 + 1 = 6$

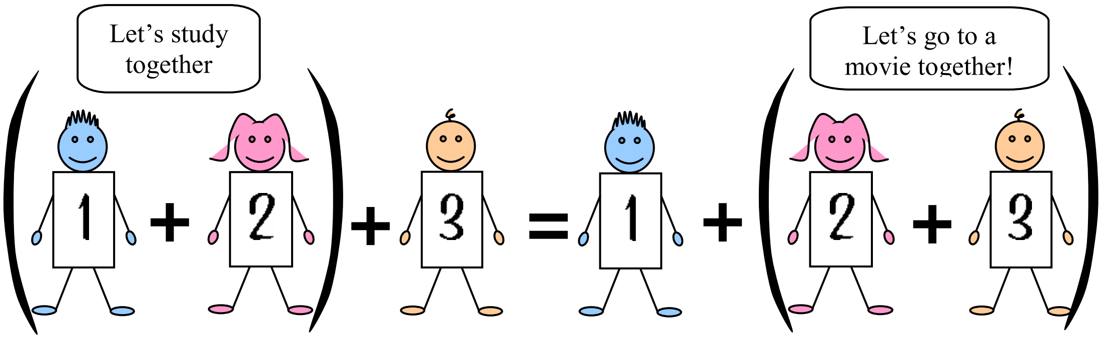
Got the idea?



Associative Property of Addition: Group Activity

Property: Addends can be added in any *group*.

BrainAid: Associative [pronounced uh-SOH-shee-uh-tiv]* comes from *associate* which means to *group* together, as in “friends like to associate with each other.” Associated addends are grouped inside of parentheses. Operations inside of parentheses are performed first.



$$(1 + 2) + 3 = 1 + (2 + 3)$$
$$3 \quad + 3 = 1 + \quad 5$$
$$6 = 6$$

* See Pronunciation Guide on page 4.

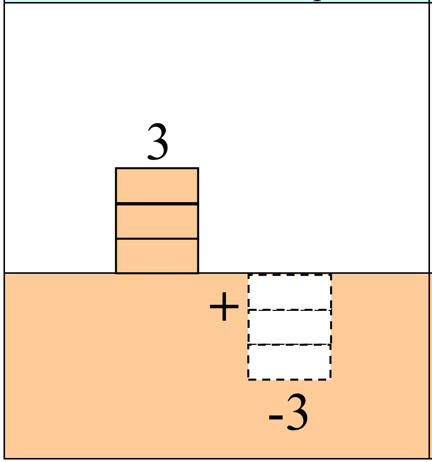
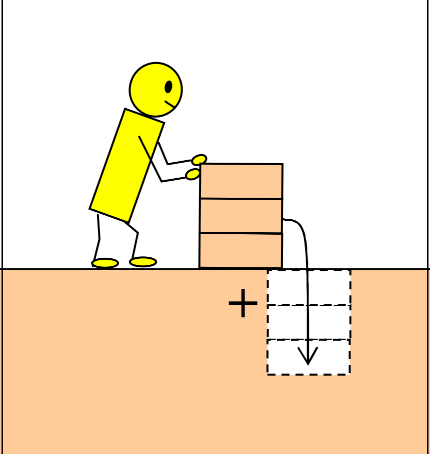
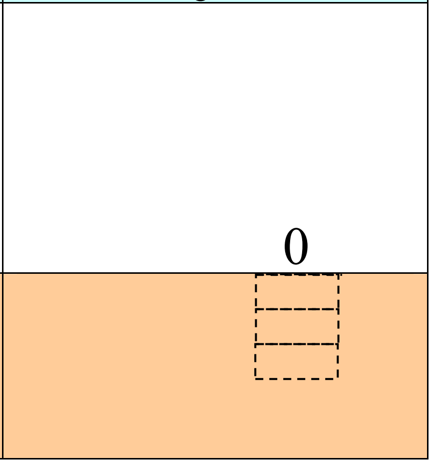
Additive Inverse: Matter meets Antimatter

Definition: An Additive Inverse has the same value but the opposite sign of an addend.

Negation: To create an additive inverse, you negate [nih-GAAT or NAA-gaat] an addend by placing a negative sign in front of it.

Property: An addend plus its inverse equals zero.

BrainAid: *Inverse* means opposite. Like matter and antimatter, inverses cancel each other out.

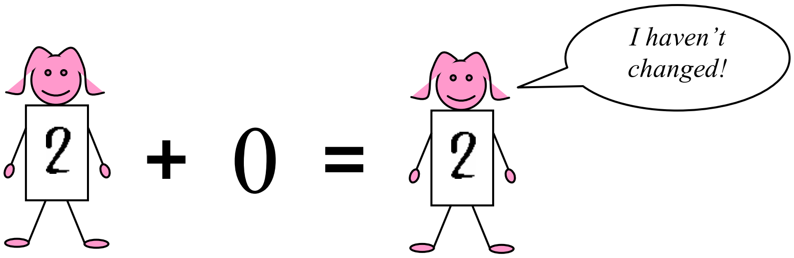
The pile is as tall as the hole is deep.	Push the pile into the hole.	The pile and hole are gone. Nothing remains.
		

Additive Identity Element: Zero Influence

Definition: The number zero is the Additive Identity Element.

Property: An addend plus zero equals the addend.

BrainAid: If you add something (e.g., wig) to your head or face, it influences your appearance. But if you add nothing (zero) to your head or face, it has zero influence—your identity remains the same.



Addition Layouts

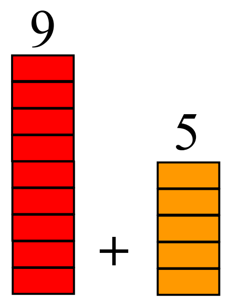
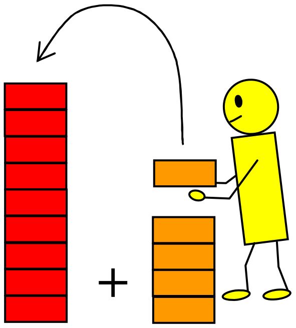
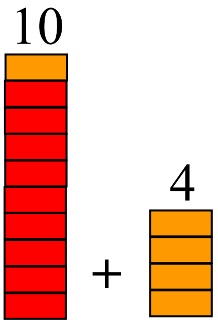
Addend + Addend = Sum	**Addend** **+ Addend** **Sum**

Mental Addition (MA)

When pencil and paper aren't available, use your head instead!

MA: Borrow

Tip: Mentally, it's easier to add numbers that end in zero.
Trick: When one addend is close to a number that ends in zero (10, 20, 30, etc.), borrow enough from another addend to make it so; then add.

When one addend is 9 (or ends in 9)...	…borrow 1 from the other addend.	Think: 9+5 = 10+4 = 14.
		

Your turn: Mentally borrow to make one addend end in zero, then add.

19 + 8 = 20 + ___ = 27

2 + 39 = 1 + ___ = ___

49 + 4 = ___ + ___ = ___

6 + 79 = ___ + ___ = ___

More Borrowing

Borrowing 1 to make 100

99 + 32 = 100 + 31 = 131

Borrowing 2 to make 100

98 + 32 = 100 + 30 = 130

Borrowing 1 to make 1000

999 + 158 = 1000 + 157 = 1157

Your turn: Imagine piles as you mentally add by borrowing.

99 + 44 = ___

99 + 86 = ___

999 + 47 = ___

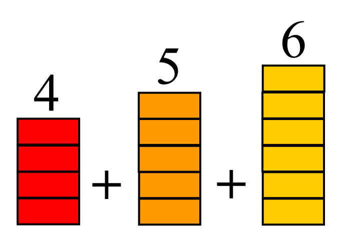
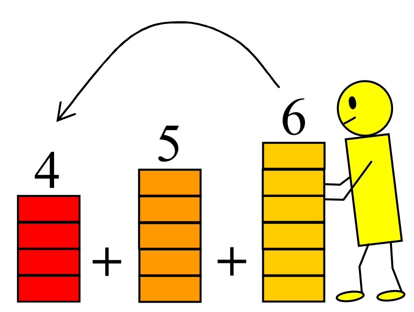
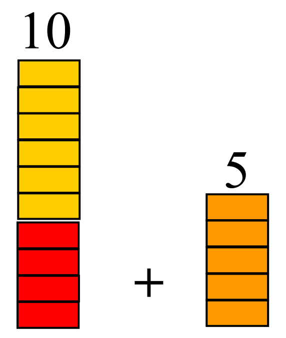
99 + 99 = ___

72 + 98 = ___

998 + 317 = ___

MA: Find 10s

Tip: Mentally, it's easier to add 10s.
Trick: Find and group addends that make 10; attach remaining addend(s).

With several addends, look for any that add to 10.	Attach those addends first. Think: 4+6 = 10	Now attach the remaining addend. Think: = 10+5 = 15
		

Your turn: Make 10s and fill in the missing numbers.

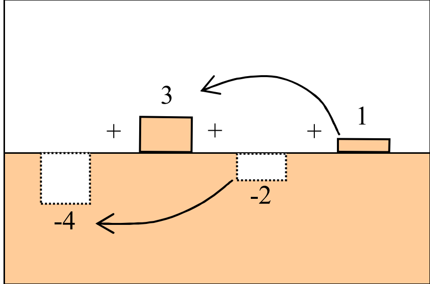
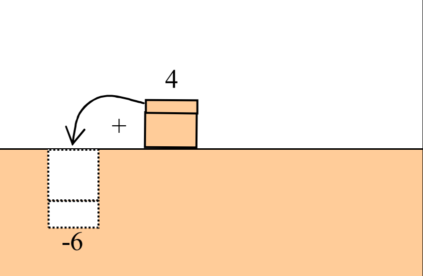
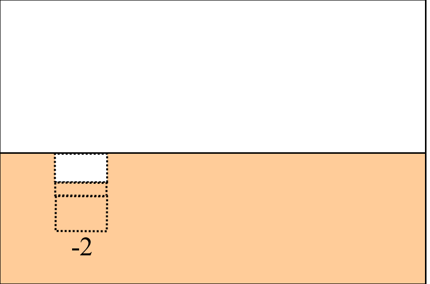
7 + 9 + 3 = 10 + __ = ___

2 + 6 + 8 + 4= __ + __ = ___

6 + 5 + 3 + 1 = __ + __ = ___

MA: Stack Signs

Tip: Mentally, it's easier to add negative and positive numbers separately.
Trick: Stack all holes; stack all piles; attach resulting hole and pile.

Stack holes with holes and piles with piles.	Attach the hole and pile.	The sum.
		

Your turn: Stack by signs, then attach the resulting hole and pile.

-3 + 4 + -6 + 3
-3 + __ + 4 + 3
-9 + __
____

6 + -5 + 2 + -1
6 + __ + __ + -1
__ + __
____

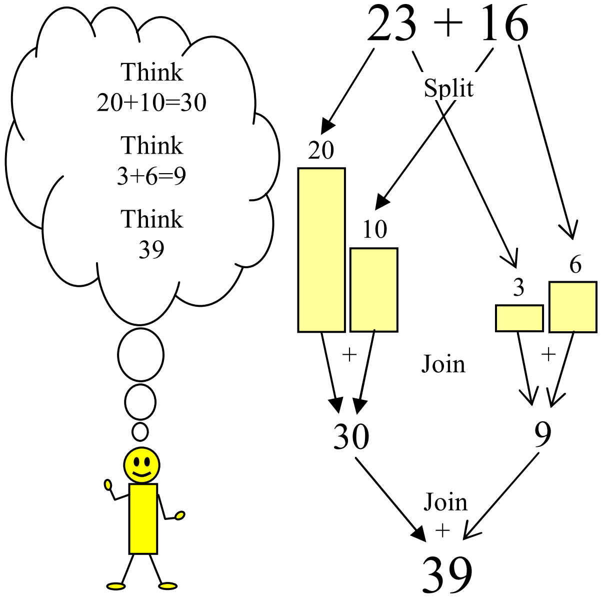
3 + -1 + -4 + 2 + 1
3 +__+__+__+ -4
__ + __
____

MA: Split & Join

Tip: Mentally, it's easier to add numbers highest-to-lowest by place value (see page 7).
Trick: Split the addends into place values (100s, 10s, 1s), then join digits starting with the highest place value, so that the final sum is already in the order you'd think or say it.

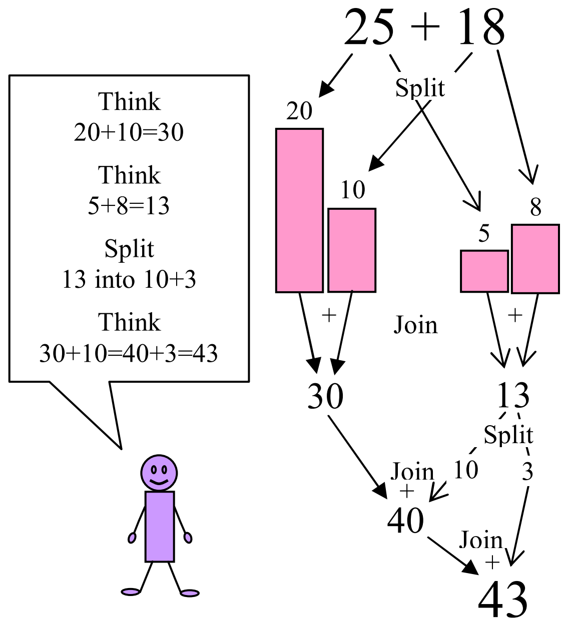
Place-Value Sum Less than 10

If the ones-place sum is less than 10, join that sum directly to the tens-place sum.



Place-Value Sum 10 or More

If the ones-place sum is 10 or more, split it into 10 + a remainder, then join each in turn.



Your turn: As you mentally add, fill in the boxes with what you are thinking.

72 + 14

70 + 10 = ____
2 + 4 = ____
_____

42
+ 53

40 + ____ = ____
2 + ____ = ____
_____

Your turn: Fill in the boxes with what you are thinking. Check your answers in the back.

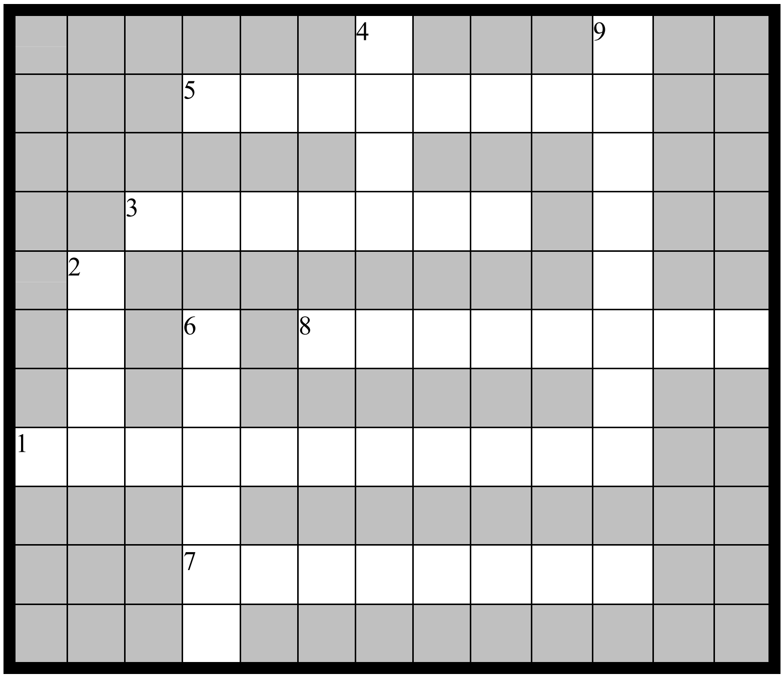
57 + 28

50 + 20 = ____
7 + 8 = ____
70 + 10 = 80 + ____
_____

96 + 77

____ + ____ = 160
____ + ____ = 13
____ + ____ = 170 + 3
_____

BrainDrain #1



Fill in the Crossword Puzzle

Across

1. The _____ property reorders addends.
3. Add the _____ addend place value first.
5. Inverse means _____.
7. The plus sign is a mathematical _____.
8. Zero is the Additive _____ Element.

Down

2. Whole numbers include _____.
4. Plus means _____.
6. A number is a _____ for a quantity or value.
9. Integers include _____ numbers.

True/False

Write T or F in the blanks.

1____ All whole numbers are integers.

2____ All integers are whole numbers.

3____ All positive integers are natural numbers.

4____ The associative property groups addends.

5____ The Additive Inverse is zero.

6____ 2nd is an example of a Cardinal number.

To succeed at math, especially Mental Math, strive to get into a RUFF frame of mind by being:

R = Relaxed
U = Uncluttered
F = Focused
F = Flowing

Slow down, breathe deeply, be calm.

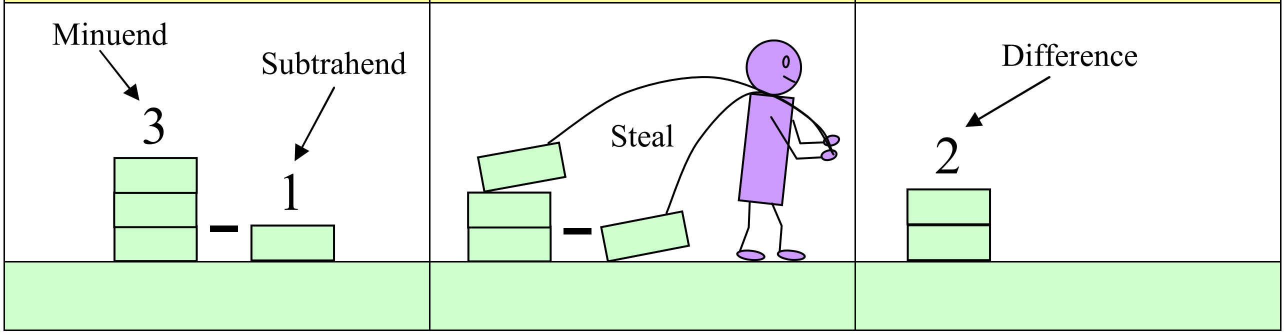
Daily Practice

The more mental addition you do, the faster and more accurate you'll be. Look for numbers to add together in the newspaper, on street signs, on license plates, etc. Or just pick numbers at random, and see if you can add them in your head. Take an extra second to visualize piles and holes, so you have a feel for the magnitude of numbers and avoid calculation errors.

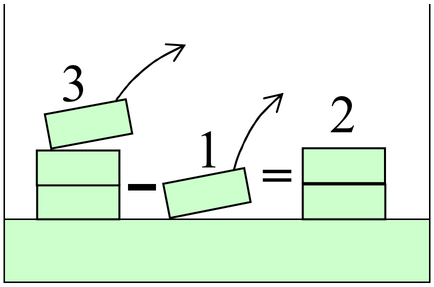
Subtraction Steals

Smaller Pile (3 – 1 = 2)

MINUEND - SUBTRAHEND	OPERATOR	DIFFERENCE
The first number is the MIN-uu-end. The <u>S</u>econd number is the <u>S</u>UB-truh-hend.	*Minus* is Latin for *less.* The minus sign says to 'steal' the subtrahend and an equal…	…amount from the <u>*min*</u>uend to di<u>*min*</u>ish it. What's left is called the *difference.*



We can redraw the subtraction this way.



Try it: Use piles and arrows to sketch the following subtraction: 2 – 1 = 1

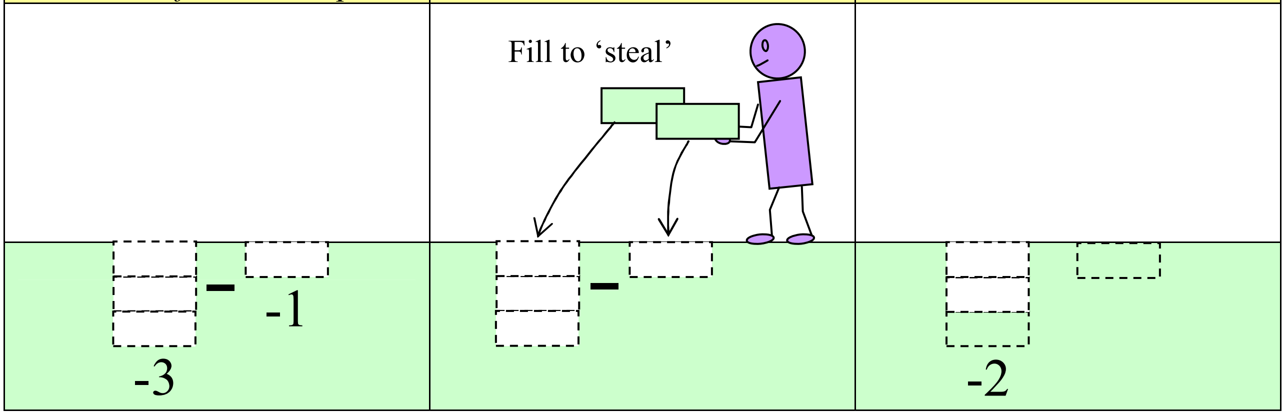
Your turn: Imagine stealing the subtrahend and an equal amount from the minuend. These are simple problems, but taking time to visualize the piles will pay off in accuracy.

$\overset{\nearrow}{4} – \overset{\nearrow}{3} =$ ___ $\quad \overset{\nearrow}{5} – \overset{\nearrow}{2} =$ ___ $\quad \overset{\nearrow}{6} – \overset{\nearrow}{3} =$ ___

$\overset{\nearrow}{7} – \overset{\nearrow}{2} =$ ___ $\quad \overset{\nearrow}{9} – \overset{\nearrow}{3} =$ ___ $\quad \overset{\nearrow}{9} – \overset{\nearrow}{6} =$ ___

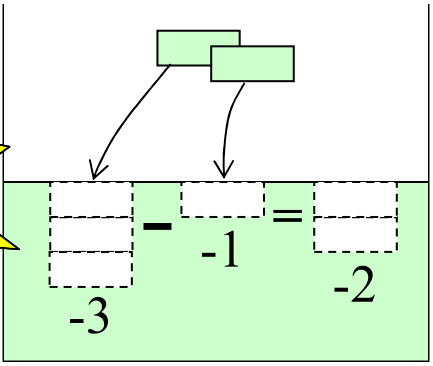
Smaller Hole (-3 – -1 = -2)

You can’t pick it up and carry it away, so how do you steal a hole? You *fill* it with a pile!	And you add the same size pile to the minuend to maintain the difference.	The subtrahend hole vanishes, and the *difference* is a smaller hole.



Written more compactly, the subtraction appears this way.

Paradox
To *subtract* a hole, you must *add* a pile!



Try it: Use holes, piles, and arrows to sketch the following subtraction: -2 – -1 = -1

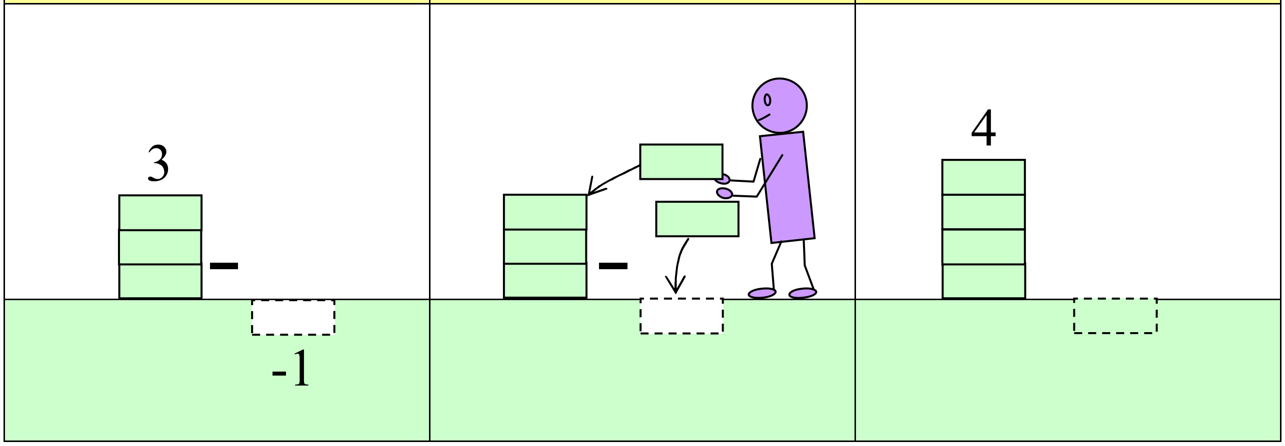
Your turn: Mentally fill in the subtrahend hole, and put an equal amount in the minuend hole.

-3 – -2 = ____ -5 – -2 = ____ -6 – -2 = ____

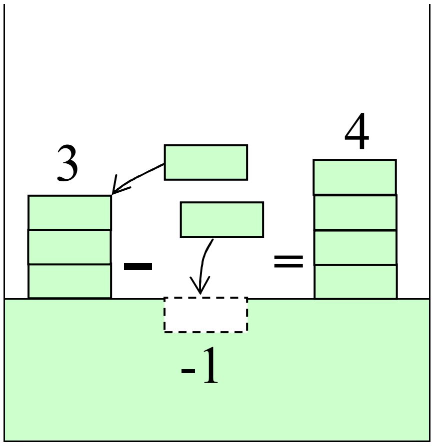
-8 – -5 = ____ -9 – -3 = ____ -9 – -7 = ____

Larger Pile (3 – -1 = 4)

As before, you must steal the subtrahend hole by filling it…	…and adding the same amount to the minuend...	...which results in a larger pile as the *difference*.



We can redraw the subtraction more compactly.



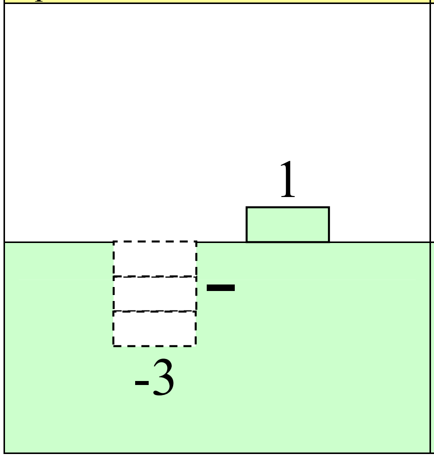
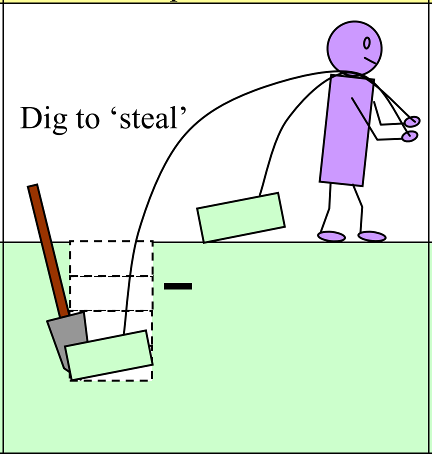
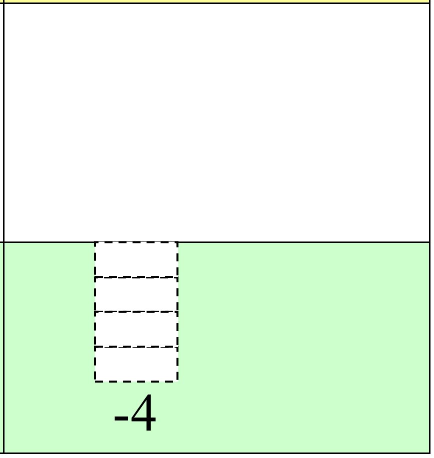
Try it: Use piles, a hole, and arrows to sketch the following subtraction: 2 – -1 = 3

Your turn: Imagine filling in the subtrahend and adding an equal amount to the minuend.

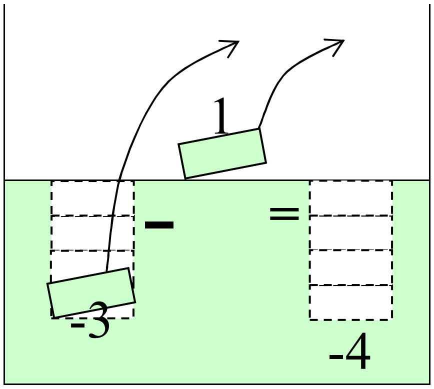
↙ ↙ 4 – -3 = ___ ↙ ↙ 5 – -1 = ___ ↙ ↙ 6 – -3 = ___

↙ ↙ 7 – -4 = ___ ↙ ↙ 9 – -3 = ___ ↙ ↙ 9 – -6 = ___

Larger Hole (-3 – 1 = -4)

You can steal the subtrahend pile, but how do you steal a pile from the minuend hole?	You *dig* the hole deep enough to remove a pile of equal size!	The resulting *difference* is a deeper hole.
		

Here's the condensed version.



Try it: Use holes, piles, and arrows to sketch the following subtraction: -2 – 1 = -3

Your turn: Mentally steal the subtrahend pile and an equal amount from the minuend hole.

$\overset{\nearrow}{-4} - \overset{\nearrow}{2} =$ ____ $\overset{\nearrow}{-5} - \overset{\nearrow}{2} =$ ____ $\overset{\nearrow}{-6} - \overset{\nearrow}{2} =$ ____

$\overset{\nearrow}{-7} - \overset{\nearrow}{3} =$ ____ $\overset{\nearrow}{-9} - \overset{\nearrow}{3} =$ ____ $\overset{\nearrow}{-9} - \overset{\nearrow}{5} =$ ____

Properties of Subtraction

Subtraction lacks the common properties of addition.

No Commutative Property of Subtraction

With addition you can change the number order, but *not* with subtraction.
Example: 1 – 2 = -1 is not the same as 2 – 1 = 1.
BrainAid: Governors do *not* commute the sentences of negative prisoners.

No Associative Property of Subtraction

With addition, you can arrange numbers in any group, but *not* with subtraction.
Example: (1 – 2) – 3 = -1 – 3 = -4 is not the same as 1 – (2 – 3) = 1 – -1 = 2
BrainAid: You should *not* associate with negative people.

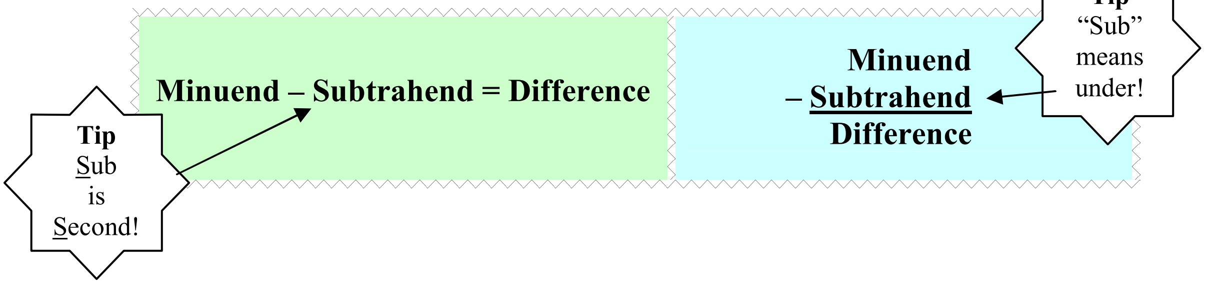
No Subtractive Inverse

With addition, inverses cancel each other out and make zero, but *not* with subtraction.
Example: 3 and -3 are inverses, but 3 – -3 = 6. It does *not* equal zero.

Subtractive Identity Element?

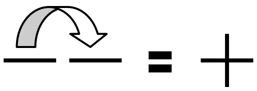
With addition, 2 + 0 = 2 and 0 + 2 = 2. With subtraction, this only works if 0 is the subtrahend.
Example: 2 – 0 = 2 but 0 – 2 = -2

Subtraction Layouts



Subtracting a Negative Makes A Positive

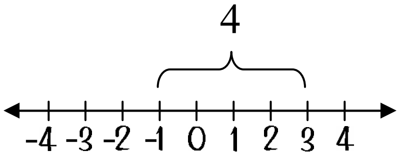
When you see a double negative, imagine rotating the first minus sign and placing it over the second to create a plus sign.



Example: 3 – –1 = 3 + 1 = 4

Number Line Difference

A number line is another way to show the *difference* between positive and negative numbers.
Example: 3 – –1 = 4, i.e., the *difference* between 3 and –1 is 4 (*i.e.* is Latin for *id est* which means "that is").



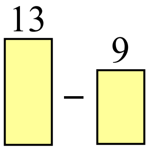
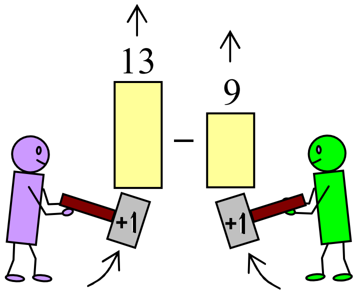
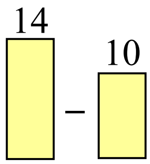
Mental Subtraction (MS)

MS: Bump

Tip: Mentally, it's easier to subtract numbers that end in zero.

Trick: When the subtrahend is close to a number that ends in zero (10, 20, 30, etc.) bump it up or down to make it so. Likewise, bump the minuend up or down the same amount to maintain the difference between the two, then subtract.

Bump Up

When the subtrahend is *below* a number that ends in zero...	…bump both it and the minuend *up*.	Think: 13 – 9 = 14 – 10 = 4
		

Your turn: Fill in the blanks as you bump up both numbers, then subtract.

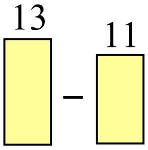
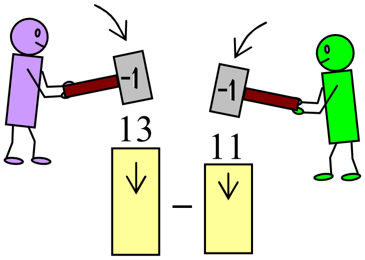
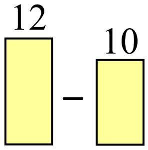
$\overset{\uparrow}{17} - \overset{\uparrow}{9} = 18 - \underline{\quad} = 8$

$\overset{\uparrow}{42} - \overset{\uparrow}{29} = \underline{\quad} - 30 = \underline{\quad}$

$\overset{\uparrow}{76} - \overset{\uparrow}{39} = \underline{\quad} - \underline{\quad} = \underline{\quad}$

$\overset{\uparrow}{76} - \overset{\uparrow}{38} = \underline{\quad} - \underline{\quad} = \underline{\quad}$

Bump Down

When the subtrahend is *above* a number that ends in zero...	…bump both it and the minuend *down*.	Think: 13 – 11 = 12 – 10 = 2
		

Your turn: Fill in the blanks as you bump both numbers down, then subtract.

$\overset{\downarrow}{43} - \overset{\downarrow}{11} = 42 - \underline{\quad} = 32$

$\overset{\downarrow}{50} - \overset{\downarrow}{31} = \underline{\quad} - 30 = \underline{\quad}$

$\overset{\downarrow}{70} - \overset{\downarrow}{41} = \underline{\quad} - \underline{\quad} = \underline{\quad}$

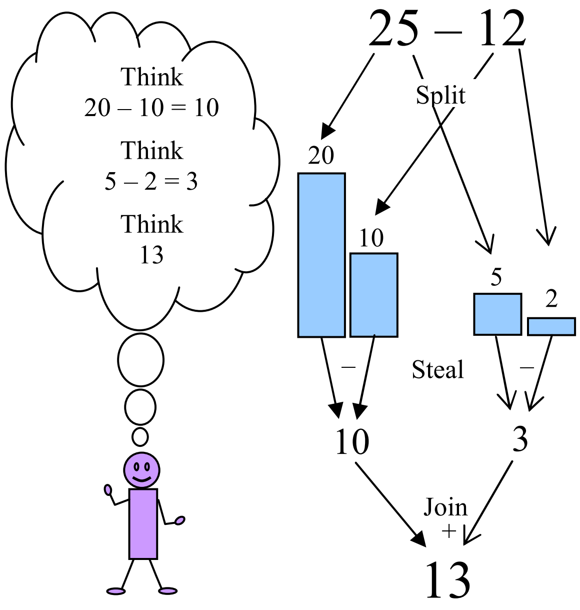
$\overset{\downarrow}{70} - \overset{\downarrow}{42} = \underline{\quad} - \underline{\quad} = \underline{\quad}$

MS: Split & Steal

Tip: Mentally, it's easier to subtract numbers highest-to-lowest by place value (see page 7).
Trick: Split the minuend and subtrahend into place values (100s, 10s, 1s), then subtract digits starting with the highest place value, so that the difference is already in the order you'd think or say it.

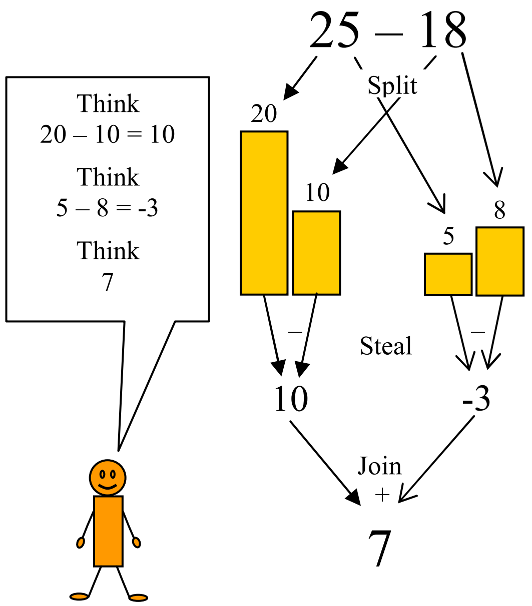
All Minuend Digits Larger

If all minuend place values are equal to or greater than the subtrahend values beneath, join resulting piles.



Some Minuend Digits Smaller

If some minuend place values are less than the subtrahend values beneath, join the resulting pile and hole.



Your turn: As you mentally subtract, fill in the boxes with what you are thinking.

76 – 34
70 – 30 = ___
6 – 4 = ___
42

48
– 25
___ – ___ = 20
___ – ___ = 3
_____

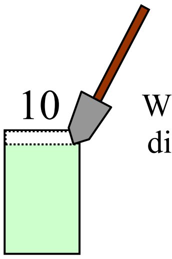
Your turn: Fill in the boxes with what you are thinking. Check your answers in the back.

76 – 38
70 – 30 = ___
6 – 8 = ___
_____

48
– 29
___ – ___ = 20
___ – ___ = -1
_____

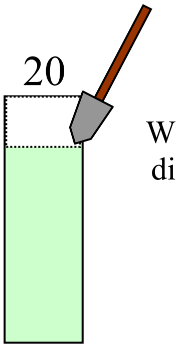
MS: Dig Pile

Tip: Mentally, when subtracting, it's quicker to dig a hole into a pile.
Trick: Rather than pushing a pile into a hole, dig directly into the pile.



What remains when you dig a 1-deep hole into a 10-high pile?

A 9-high pile.



What remains when you dig a 4-deep hole into a 20-high pile?

A 16-high pile.

Memorize The Hole/Pile Pairs

Starting with a 10-high pile:

If you dig -1, you're left with 9.
If you dig -2, you're left with 8.
If you dig -3, you're left with 7.
If you dig -4, you're left with 6
If you dig -5, you're left with 5

If you dig -6, you're left with 4
If you dig -7, you're left with 3
If you dig -8, you're left with 2
If you dig -9, you're left with 1

Try it

Starting with a 40-high pile:

If you dig -3, you're left with ____.

If you dig -8 you're left with ____.

Answers: 37, 32

Starting with a 100-high pile:

If you dig -6, you're left with ____.

If you dig -30 you're left with ____.

Answers: 94, 70

Dig Pile With Split & Steal	
43 – 14 Think: 40 – 10 = 30-high pile Dig a (3 – 4) 1-deep hole = **29**	**132 – 54** Think: 100 – 0 = 100-high pile Dig a (30 – 50) 20-deep hole = 80 Dig a (2 – 4) 2-deep hole = **78**

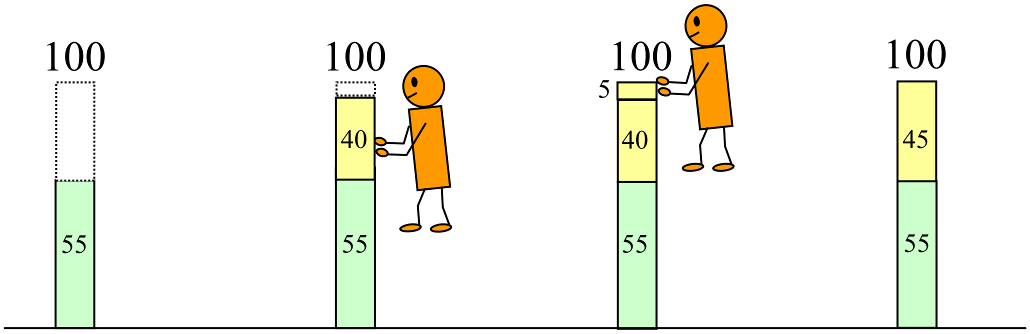
Your turn: Dig piles to fill in the blanks.

94 – 78 90 – 70 = ____-high pile Dig a (___–___) 4-deep hole = _____	**153 – 68** ____–____= 100-high pile Dig a (____–____) 10-deep hole = _____ Dig a (3 – 8) ___-deep hole = ____.

MS: Fill Up

Tip: Mentally, when a minuend ends in zeros (100, 1000, etc.), it's easier to compute the difference by filling up the gap between the subtrahend and the minuend.

Trick: Add enough to the subtrahend to make it reach the minuend.



To fill up from 55 to 100...

Add 40 to make 95.

Add 5 to make 100.

The difference of 45 fills the gap.

Here are the rules for filling up, starting with the highest place value.

Here's an example with larger numbers.

1	0	0	MINUEND	2	0	0	0
0	4	5	Fill Up	1	6	4	3
	5	5	SUBTRAHEND		3	5	7
Fill to 1 less than minuend	Fill to 9	Fill to 10	Rule	Fill to 1 less than minuend	Fill to 9		Fill to 10

Try it: Starting with the highest place value, fill up the gap.

4	0	0	0
1	2	4	6
Fill to 1 less than minuend	Fill to 9		Fill to 10

Answer: 2754

Why it works

Partially filling each place value from highest to lowest creates an answer in the order that you'd think or say it. The final step, filling the 1s-place to 10 forces a 1 to be carried to the 10s-place which fills it, forcing a 1 to be carried to the 100s-place, and so on until this cascading effect fills up the entire gap.

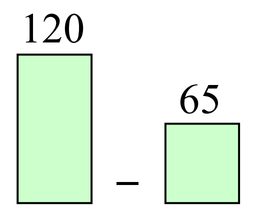
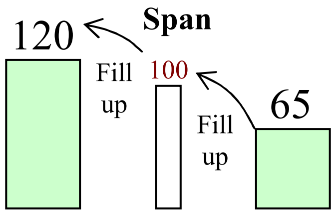
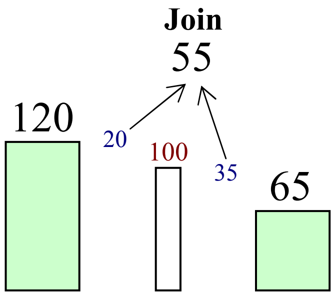
Your turn: Fill up to find the differences.

100 – 37 = ___	200 – 58 = ___	200 – 132 = ___
1000 – 374 = ___	2000 – 582 = ___	2000 – 1329 = ___

MS: Span & Join

Tip: Mentally, when the minuend is greater than and the subtrahend is less than a number that ends in zeros (100, 1000, etc.), it's easier to find the difference by spanning both gaps.

Trick: Insert the zero-ending number between the minuend and the subtrahend. Fill both gaps and add.

The minuend is above 100; the subtrahend is below 100.	Think: 65 to 100 = 35. Think: 100 to 120 = 20.	Think: 35 + 20 = 55, which is the *difference*.
		

Your turn: Fill in the blanks as you span and join the gaps.

140 – 90 = ___

116 – 40 = ___

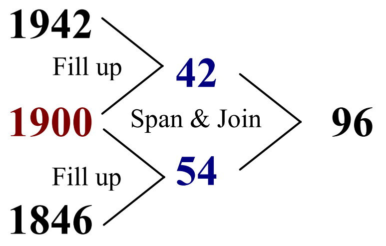
250 –175 = ___

325 – 195 = ___

Century Span

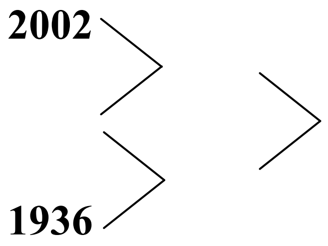
This technique is especially useful for figuring the span in years when a century mark is crossed.

What was the lifespan of someone who was born in 1846 and died in 1942?



Your turn: Span the century and fill in the blanks.

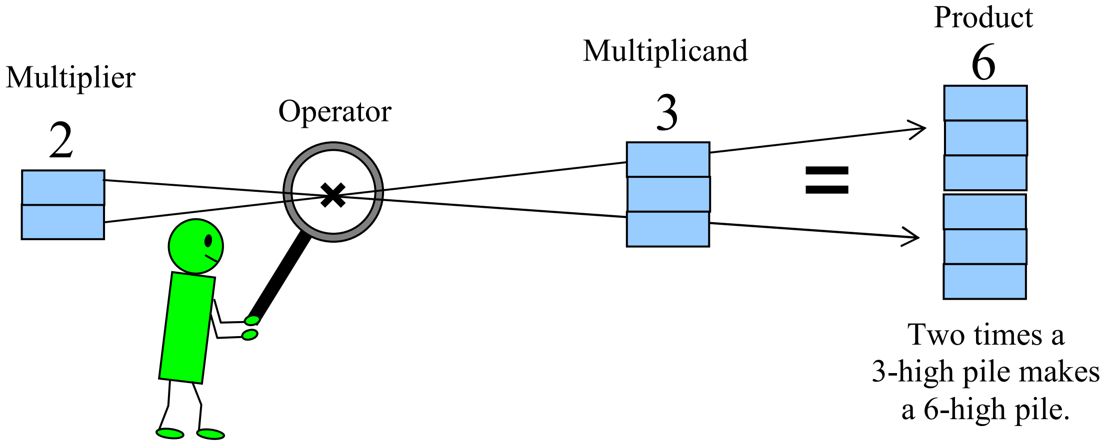
Born: 1936 Died: 2002. How many years?



Multiplication Magnifies

Larger Pile (2 × 3 = 6)

MULTIPLIER	OPERATOR	MULTIPLICAND	PRODUCT
Think of the multipl*ier* as the magnif*ier*.	Imagine the *times* symbol, ×, etched into a magnifying glass.	The multiplicand is the number that can be magnified.	Think of a product made in a factory from smaller parts.



Two times a 3-high pile makes a 6-high pile.

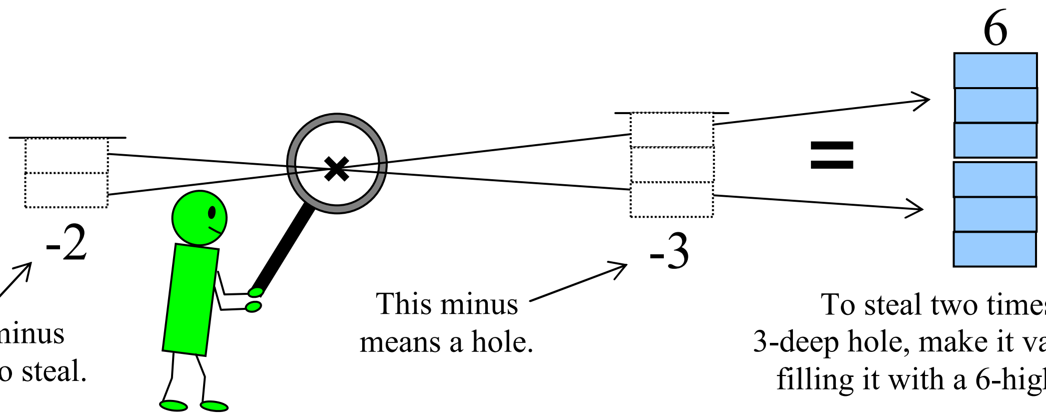
Your turn: Imagine magnifying a pile.

$\overrightarrow{2 \times 4} =$ ___ $\overrightarrow{3 \times 3} =$ ___ $\overrightarrow{4 \times 3} =$ ___

$\overrightarrow{5 \times 3} =$ ___ $\overrightarrow{2 \times 7} =$ ___ $\overrightarrow{1 \times 9} =$ ___

Larger Pile (-2 × -3 = 6)

MULTIPLIER	OPERATOR	MULTIPLICAND	PRODUCT



This minus means to steal.

This minus means a hole.

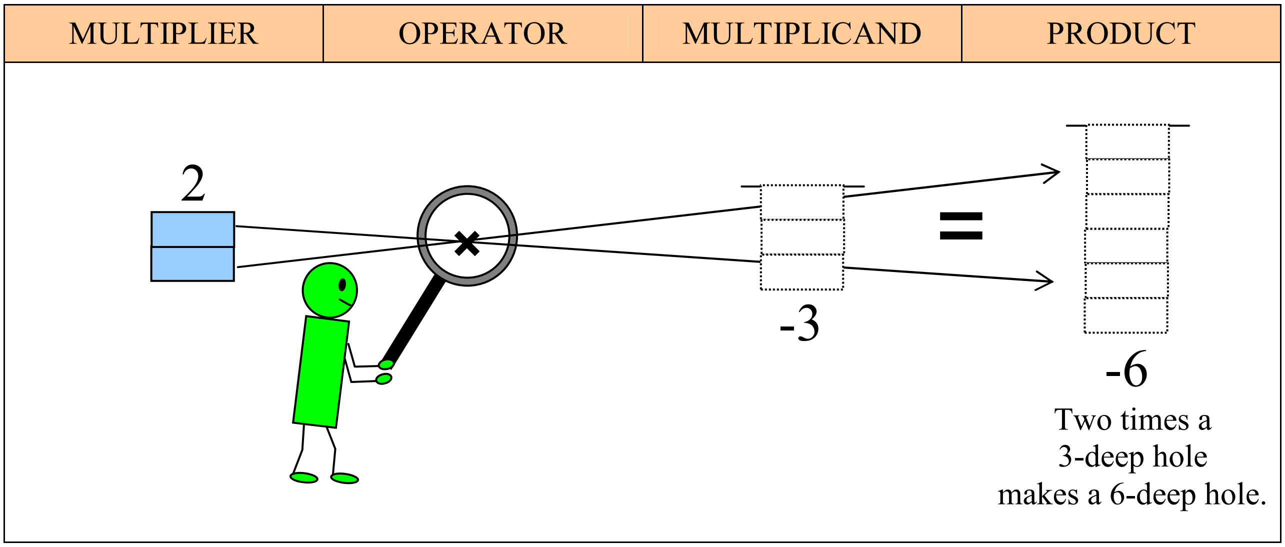
To steal two times a 3-deep hole, make it vanish by filling it with a 6-high pile.

Your turn: Imagine stealing a magnified hole by filling it with a pile.

$\overrightarrow{-2 \times -5} =$ ___ $\overrightarrow{-3 \times -4} =$ ___ $\overrightarrow{-4 \times -2} =$ ___

$\overrightarrow{-5 \times -4} =$ ___ $\overrightarrow{-2 \times -6} =$ ___ $\overrightarrow{-1 \times -9} =$ ___

Larger Hole (2 × -3 = -6)



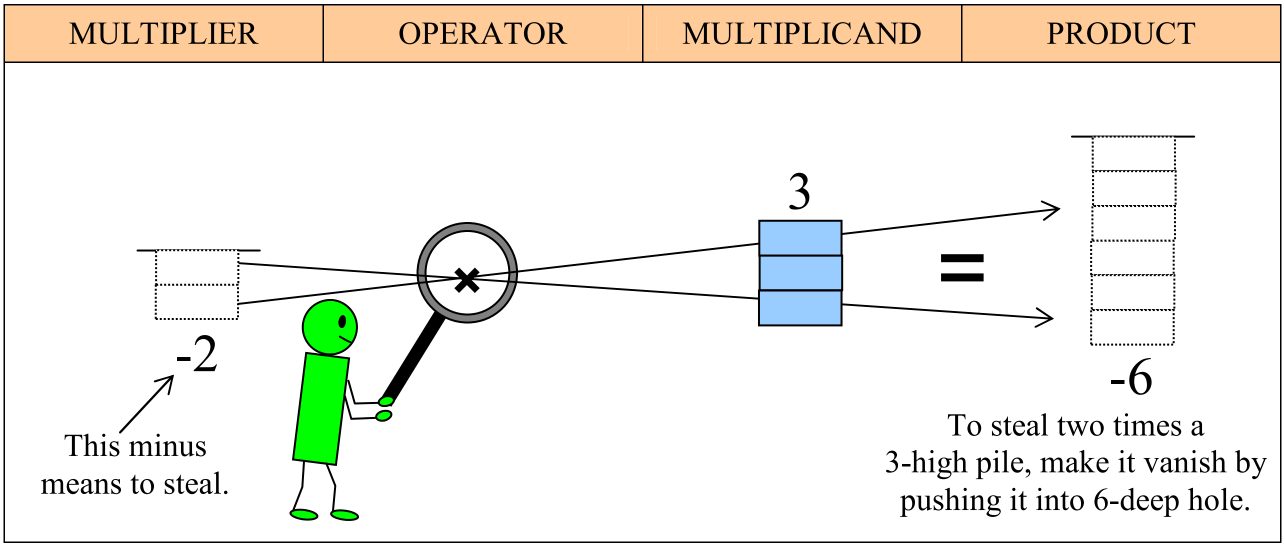
Two times a 3-deep hole makes a 6-deep hole.

Your turn: Imagine magnifying a hole.

$\overrightarrow{2 \times -4} =$ ____ $\quad$ $\overrightarrow{3 \times -2} =$ ____ $\quad$ $\overrightarrow{4 \times -3} =$ ____

$\overrightarrow{3 \times -5} =$ ____ $\quad$ $\overrightarrow{2 \times -7} =$ ____ $\quad$ $\overrightarrow{1 \times -9} =$ ____

Larger Hole (-2 × 3 = -6)



This minus means to steal.

To steal two times a 3-high pile, make it vanish by pushing it into 6-deep hole.

Your turn: Imagine stealing a magnified pile by pushing it into a hole.

$\overrightarrow{-2 \times 5} =$ ____ $\quad$ $\overrightarrow{-3 \times 4} =$ ____ $\quad$ $\overrightarrow{-3 \times 2} =$ ____

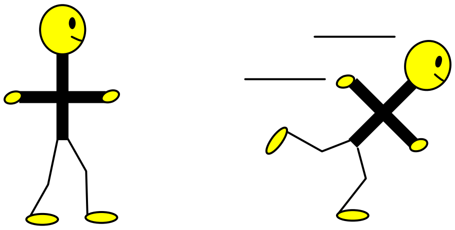
$\overrightarrow{-5 \times 4} =$ ____ $\quad$ $\overrightarrow{-2 \times 8} =$ ____ $\quad$ $\overrightarrow{-1 \times 9} =$ ____

Properties of Multiplication

Multiplication has many of the same properties as addition.

Multiplication = *Fast* Addition

While Addition saunters along at a leisurely pace, Multiplication zips on by. Multiplication is fast if you've memorized your times tables.



$$5 + 5 + 5 + 5 = 4 \times 5$$

Multiplication Variations

Multiplication has several operators...

2 × 3
2 • 3
2 * 3

...or no operators at all!

2(3)
(2)(3)
2a
ab

(a and b can be any numbers)

Commutative Property of Multiplication: Changing Order

Property: Multipliers can be multiplied in any *order*.

BrainAid: See the Commutative Property of Addition BrainAid. Also, think of the "mu" in com*mu*tative as meaning *mu*ltiplication. And while you're at it, think of the "at" in commut*at*ive as meaning *ad*dition, since only addition and multiplication are commutative—subtraction and division are not!

Multiplicand = Multiplier

Typically, the multiplier comes first and magnifies the multiplicand. Since the commutative property says the multiplier and multiplicand can change places, we can use "multiplier" to refer to *either* number multiplied.

$2 \times 3 = 6$
$3 \times 2 = 6$

Changing the order does not change the product.

With 3 multipliers:

$2 \times 3 \times 4 = 24$
$2 \times 4 \times 3 = 24$
$3 \times 2 \times 4 = 24$
$3 \times 4 \times 2 = 24$
$4 \times 2 \times 3 = 24$
$4 \times 3 \times 2 = 24$

Associative Property of Multiplication: Group Activity

Property: Multipliers can be multiplied in any *group*.

BrainAid: See the Associative Property of Addition BrainAid.

$$\begin{aligned} \mathbf{(2 \times 3) \times 4} &= \mathbf{2 \times (3 \times 4)} \\ \mathbf{6 \quad \times 4} &= \mathbf{2 \times \quad 12} \\ \mathbf{24} &= \mathbf{24} \end{aligned}$$

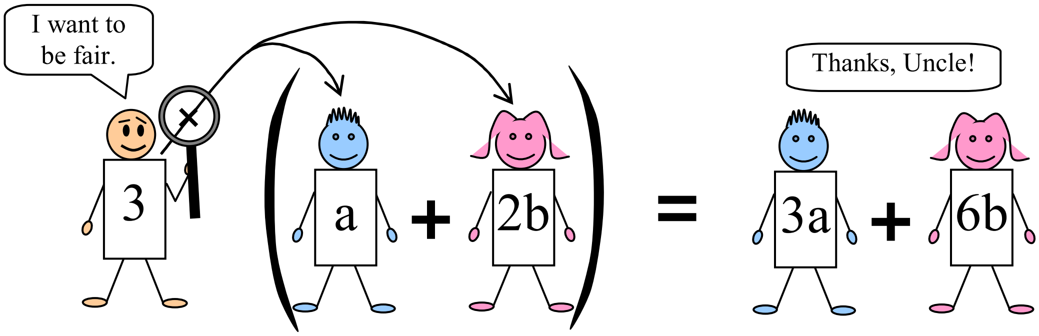
Distributive Property of Multiplication: A Rich Uncle

Property: A multiplier outside a set of parentheses magnifies each *added* or *subtracted* item that is inside the parentheses. Pronounced [di-STRI-byu-tiv].

$$2\,(a + b - c) = 2a + 2b - 2c$$

a, b, and c can be any numbers.

BrainAid
Imagine a rich uncle *distributing* his wealth to each of his nieces and nephews. Being scrupulously fair, he equally magnifies whatever value each already has.



Your turn: Be the rich uncle and distribute your wealth fairly as you fill in the blanks.

$3\,(4 + 2b) = 12 + \underline{\qquad}$

$4\,(6a - 5) = \underline{\qquad} - \underline{\qquad}$

$\underline{\quad}\,(a + 2) = 4a + 8$

$2\,(\underline{\qquad} + \underline{\qquad}) = 6a + 10b$

TRAP!

The Distributive Property does *not* apply to *divided* or *multiplied* items inside the parentheses.

This means "not" equal.

$$2\,(a \div b) \neq 2a \div 2b$$

In cases like this, the multiplier magnifies only the *first* item.

$$2\,(a \div b) = 2a \div b$$

(Try both situations with a = 6 and b = 3)

Multiplicative Inverse (Reciprocal): I Flip For You

Definition: A Multiplicative [mul-tih-PLIK-uh-tiv] Inverse is the *reciprocal* [ree-SIH-proh-kuhl] of a multiplier, which is essentially the number flipped upside down. For example:

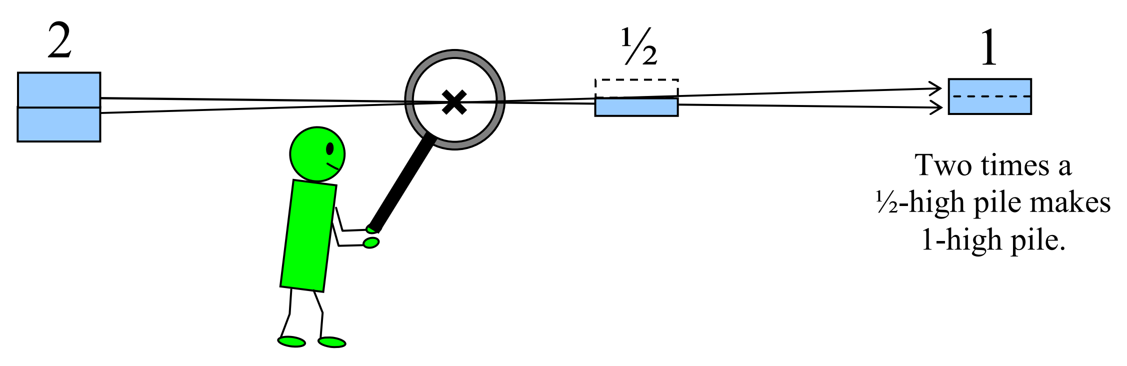
- If the multiplier is **a**, its reciprocal is **1/a**.
- If the multiplier is **-a**, its reciprocal is **-1/a**.
- If the multiplier is **1/a**, its reciprocal is **a/1** or just **a.**
- If the multiplier is **a/b**, its reciprocal is **b/a**.

a and **b** can be any numbers.

Property: A multiplier times its reciprocal equals 1; e.g., **a** × **1/a** = **1**.

BrainAid: When a boy meets a girl he likes, he 'flips' for her. If she reciprocates (returns) his feelings, they fall in love and marry, becoming one.

MULTIPLIER	OPERATOR	RECIPROCAL	PRODUCT



Two times a ½-high pile makes 1-high pile.

Your turn: Create reciprocals for each number.

4 ___ -2 ___ 1/5 ___

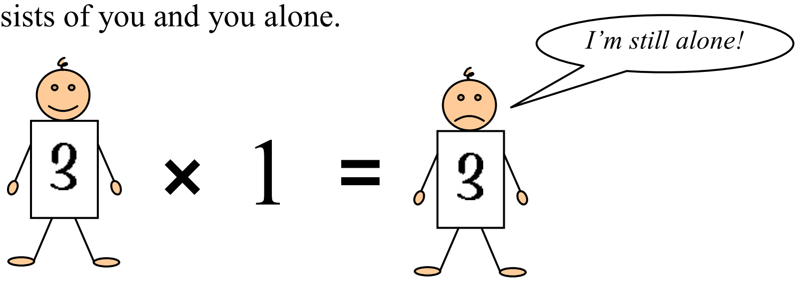
5 ___ -7 ___ 5/6 ___

Multiplicative Identity Element: One is the Loneliest Number

Definition: The number 1 is the Multiplicative Identity Element.

Property: A multiplier times 1 equals the multiplier.

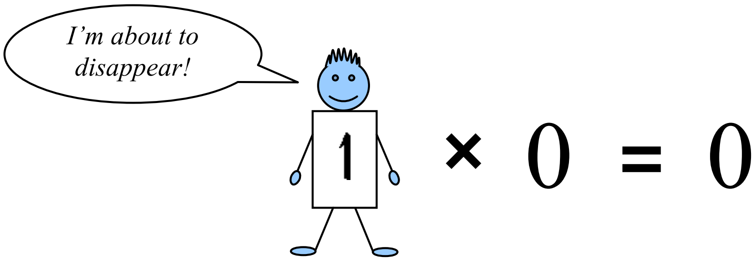
BrainAid: Your identity consists of you and you alone.



Multiplicative Property Of Zero: Makin' Nothin'

Property: A multiplier times 0 equals 0.

BrainAid: Somethin' times Nothin' leaves Nothin'.

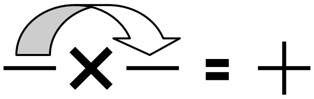


Multiplication Layouts

Multiplier × Multiplicand = Product	**Multiplicand** **× Multiplier** **Product**

Two Multiplied Negatives Make A Positive

When you multiply two negative numbers, imagine rotating the first minus sign and placing it over the second minus sign to create a plus sign for the product.



Example: –3 × –1 = 3 × +1 = +3

Multiples & Factors

A *multiple* is another name for product. A *factor* is another name for a multiplier.

Multiples are *products* created by multiplying a base number times a series of numbers.

Base × Number = Multiple

Example: 2 × 4 = 8 (8 is a multiple of base 2 and the number 4).

Your turn: Fill in the blanks in this Multiples Table.

Number Series

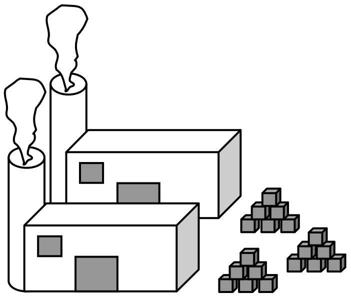
	×	2	3	4	5	6	
Base	**2**	4		8		12	**Multiples of 2**
Base	**3**		9		15		**Multiples of 3**
Base	**4**	8		16		24	**Multiples of 4**

Factors are *multipliers* that combine to make products.

Factor × Factor = Product

Example: 2 × 4 = 8 (2 and 4 are factors of 8).

BrainAid: Factories make products. Factors make products.



2 × 2 = 4

2 × 3 = 6

BrainAids

Multiples = Products
Factors = Multipliers

MPs (Military Police)
listen to FM radios.

Multiples = More*
because they're
greater than their factors.

Factors = Fewer*
because they're
less than their multiples.

* providing the factors
are positive integers.

Composite Factors

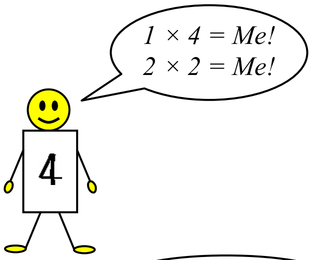
Integers divisible by 1 and themselves, and at least one other number.

Do you remember the definition of an integer? Check your answer on page 6.

Example: 4 is divisible by 1, 4, and 2.

Partial List: …-8, -6, -4, 4, 6, 8, 9, 10, 12, 14…

BrainAid: Composites are composed of many numbers.



Prime Factors

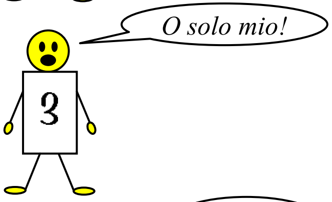
Integers divisible by 1 and themselves only.

Example: 3 is divisible by 1 and 3 only.

Partial List: …-7, -5, -3, -2, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31…

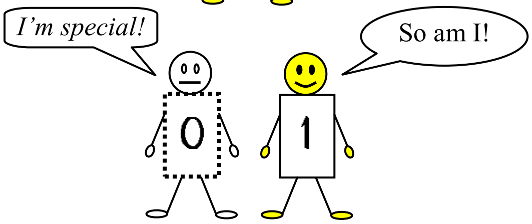
BrainAid: Primes, like prima donnas, prefer to work alone.

Fact: 2 is the only *even* prime factor.



0 and 1

By definition, 0 and 1 are neither prime nor composite.



Factoring Tricks & Trees

Noun or Verb or Both?
As a noun, *factor* means multiplier or divisor. As a verb, *factor* means to find the multipliers or divisors.

Factoring is the process of finding a product's factors.

To factor means to extract the multipliers that form a product.

Thinking in reverse, factors are also the divisors of a product.

Factoring Tricks

A product is evenly* divisible by a factor of:

2—If the product is even (i.e., ends in 0, 2, 4, 6, or 8).
3—If the sum of the product's digits is a multiple of 3 (321: 3+2+1 = 6).
4—If the product's last 2 digits are a multiple of 4 (316).
5—If the product ends in 0 or 5 (765).
6—If the product fits the tricks for both 2 and 3 above (462: 4+6+2 = 12).
7—If the product's 1st digits minus (2 × the last digit) is 0 or multiple of 7 [112: 11–(2×2) = 11 – 4 = 7].
8—If the product's last 3 digits are 000 or a multiple of 8 (2104).
9—If sum of the product's digits is a multiple of 9 (864: 8+6+4 = 18).

* Technically, *every* number is divisible by *every* number (except 0), but may not be exactly so; e.g., 10÷4 = 2½

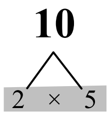
Your turn: Answer Yes or No and tell why **5580** is or is not evenly divisible by 4, 5, 6, and 9.

2	Yes, because 5580 is an even number
3	Yes, because 5+5+8+0 = 18 which is a multiple of 3 (18/3 = 6)
4	
5	
6	
7	No, because 558 – (2 × 0) = 558 – 0 = 558 which is not 0 or a multiple of 7 (558/7 = $79\frac{5}{7}$)
8	No, because 580 is a not a multiple of 8 (580/8 = 7¼)
9	

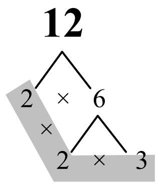
Factoring to Primes with a Factor Tree

1. Draw two branches beneath the product to be factored.
2. Divide out a prime factor and place it under the left branch with the composite under the right branch.
3. Repeat the process with the composite factor until all factors are prime. Box or shade the primes.

Factor Tree



Factor Tree



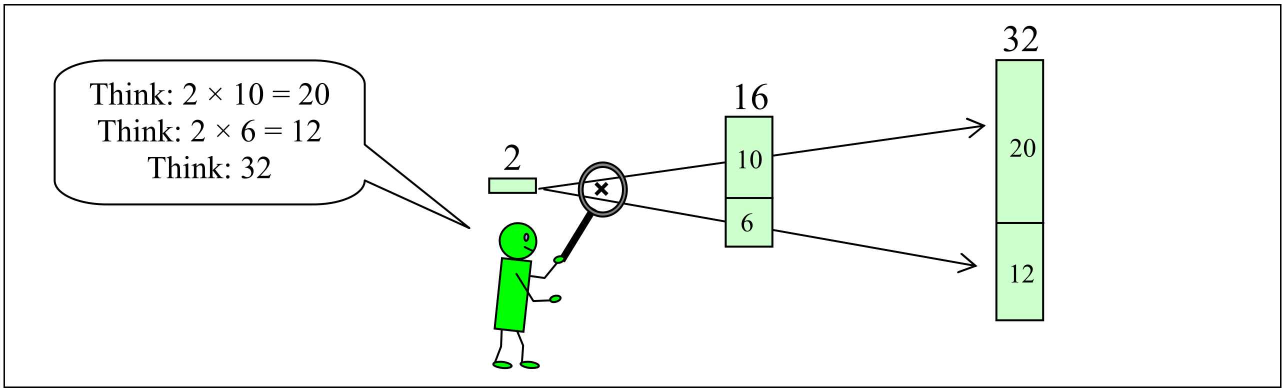
Your turn: Create Factor Trees to find prime factors for the following products:

15	16	18

Mental Multiplication (MM)

MM: Split & Double

Tip: Mentally, it’s easier to double numbers highest-to-lowest by place values.
Trick: Split the multiplicand into place values (100s, 10s, 1s), then double starting with the highest place value. Join the products and the answer is already in the order you’d think or say it.



Your turn: Mentally split and double each number.

2 × 34

2 × 30 = ____
2 × 4 = ____
_____

2 × 47

2 × ____ = ____
2 × ____ = ____
_____

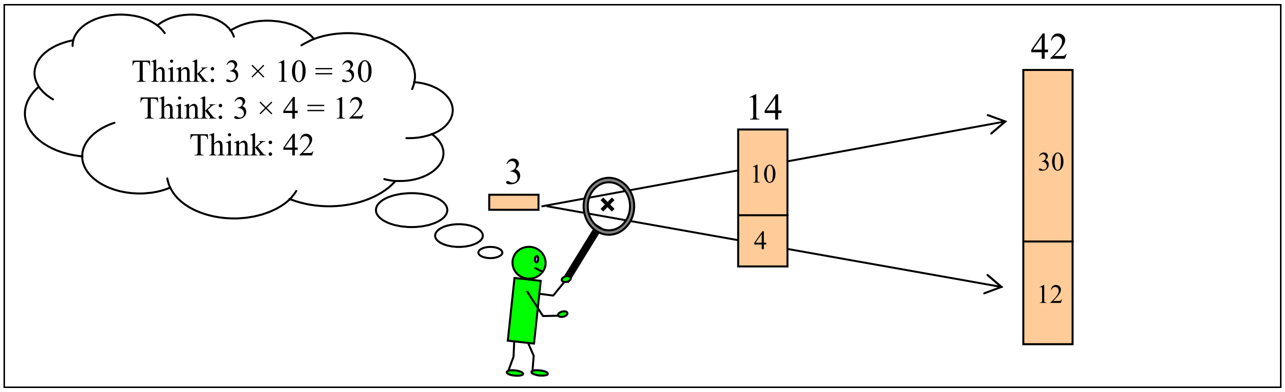
2 × 78

2 × ____ = ____
2 × ____ = ____
_____

Bonus Tip: To multiply by 4, double the number twice.

MM: Split & Magnify

Tip: Mentally, it’s easier to multiply highest-to-lowest by place values.
Trick: Same as Split & Double, but for any multiplier.



Your turn: Mentally split and magnify each number.

3 × 34

3 × 30 = ____
3 × 4 = ____
_____

6 × 47

6 × ____ = ____
6 × ____ = ____
_____

7 × 65

7 × ____ = ____
7 × ____ = ____
_____

MM: Factor & Magnify

Tip: Mentally, it's sometimes easier to factor a multiplier before multiplying.
Trick: Factor a multiplier, then regroup and multiply with the smaller factors in turn.

14 × 30
Think: (7 × 2) × 30 = 7 × (2 x 30) = 7 × 60 = 420

Your turn: Factor, regroup, and multiply.

12 × 15

(6 × ___) × 15 = 6 x (___) = ____

16 × 45

(___ × 2) × 45 = 8 x (___) = ___

MM: Multiply 5 = Half Ten

Tip: Mentally, it's sometimes easier to convert a multiplier of 5 into its equivalent ½ × 10.
Trick: Halve the number and multiply by 10 (add a zero or move the decimal point one right).

5 × 22
Think: ½ × 22 = 11 × 10 = 110

Your turn: Halve the number and multiply by 10.

5 × 24

½ × 24 = ____ × 10 = ____

5 × 68

½ ×____=____ × 10 =____

5 × 140

½ ×____=____ × 10 =____

5 × 244

½ ×____=____ × 10 =_____

MM: Multiply 25 = Quarter Hundred

Tip: Mentally, it's sometimes easier to convert a multiplier of 25 into its equivalent ¼ × 100.
Trick: Quarter the number and multiply by 100 (add two zeros or move the decimal point two right).

25 × 16
Think: ¼ × 16 = 4 × 100 = 400

Your turn: Quarter the number and multiply by 100.

25 × 24

¼ × 24 = ____ × 100 = ____

25 × 36

¼ ×____=____ × 100 =____

25 × 88

¼ ×____=____ × 100 =____

25 × 320

¼ ×____=____ × 100 =____

MM: 11 Split & Insert

Tip: Mentally, it's easier to multiply a 2-digit multiplicand by 11 using this trick.
Trick: Split the multiplicand digits apart; then insert the sum of the split digits in the center.
If the sum is 10 or more, carry and add the 1 to the left place value.

Without Carry	With Carry
11 × 45 4 (4+5) 5 **495**	**11 × 48** 4 (4+8) 8 Carry the 1 to the left place value. 4 (12) 8 **528**

Your turn: Mentally multiply by 11.

11 × 63

___ (___+___) ___

_____

11 × 79

___ (___+___) ___

___ (___) ___

_____

MM: 5-End Squared

Tip: Mentally, it's easier to square (multiply by itself) a number that ends in 5 using this trick.
Trick: Replace one of the left digits with the next higher digit and multiply. Follow with 25.

$\begin{array}{r} \overset{2}{\cancel{1}}\,5 \\ \times\ 1\,5 \\ \hline \downarrow\downarrow \\ 225 \end{array}$	$\begin{array}{r} \overset{3}{\cancel{2}}\,5 \\ \times\ 2\,5 \\ \hline \downarrow\downarrow \\ 625 \end{array}$

Your turn: Mentally square these numbers.

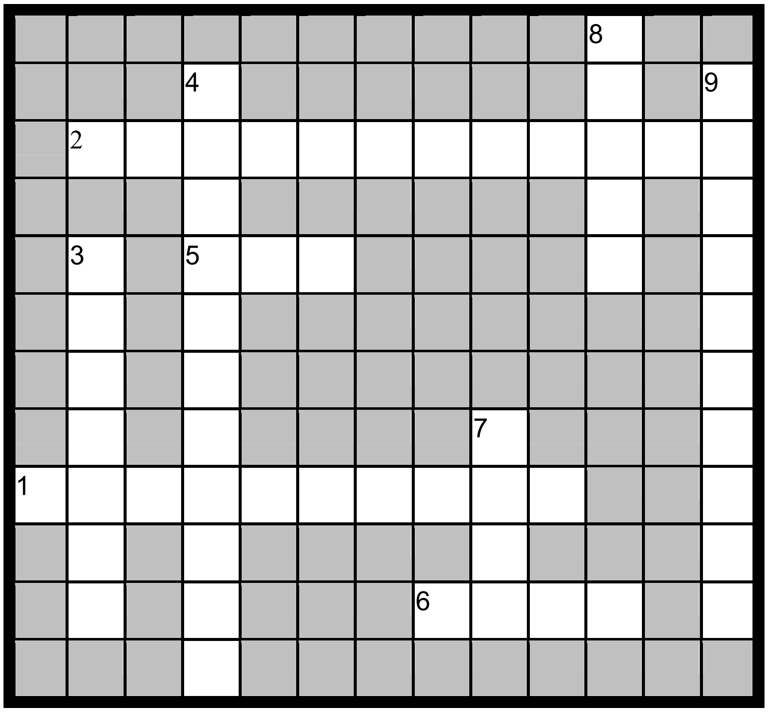
$\begin{array}{r} 3\,5 \\ \times\ 3\,5 \\ \hline \downarrow\downarrow \end{array}$

_____

$\begin{array}{r} 4\,5 \\ \times\ 4\,5 \\ \hline \downarrow\downarrow \end{array}$

_____

BrainDrain #2



Fill in the Crossword Puzzle

Across	Down
1. A factor is a _____. 2. The _____ Property multiplies items in ( ). 5. The Multiplicative Identity Element is _____. 6. "Factor" can be a _____ or a verb.	3. A multiple is a _____. 4. The ___ property groups multipliers. 7. 1 × 0 = 0 demonstrates the Property of _____ 8. A _____ factor is divisible only by 1 and itself. 9. A multiplicative inverse is also called a ______.

True/False

Write T or F in the blanks.

1____ The commutative property holds for subtraction.

2____ The associative property holds for subtraction.

3____ -6 is a reciprocal of 6.

4____ A factor is less than a multiple.

5____ 57 is a prime factor.

Speed vs. Accuracy

Taking time to visualize piles and holes may slow you down a bit, but consider this:

It doesn't matter how fast you calculate a problem, if you get it *wrong!*

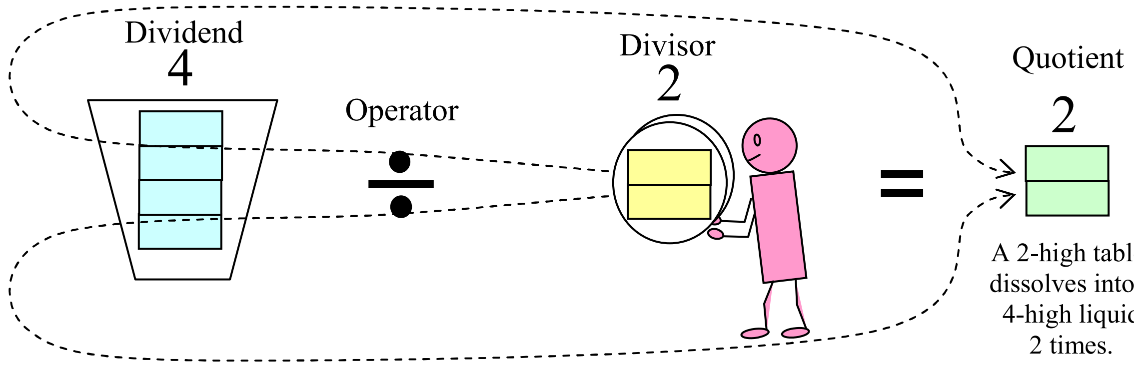
Daily Practice

Continue to seek out numbers in newspaper and magazine articles, on license plates, and street signs. Practice subtracting dates to see how many years have elapsed. Practice doubling numbers, multiplying by 5, 25, 11, etc. Challenge yourself! The more you practice, the faster and more accurate you'll be.

Division Dissolves

Smaller Pile (4 ÷ 2 = 2)

DIVIDEND	OPERATOR	DIVISOR	QUOTIENT [KWO-shunt]
Imagine a bucket of liquid.	The ÷ sign means to dissolve the tablet into the liquid.	Imagine a tablet that dissolves as many times as it fits.	The result is the *quotient* which is Latin for *how many times*.



A 2-high tablet dissolves into a 4-high liquid 2 times.

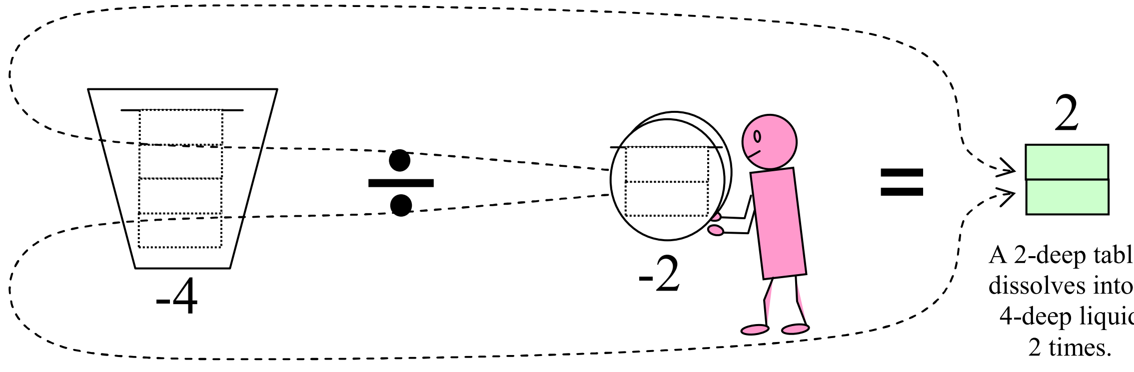
Your turn: Dissolve the positive tablet into the positive liquid as many times as it fits.

6 ÷ 3 ___ 6 ÷ 2 ___ 10 ÷ 2 ___

8 ÷ 2 ___ 8 ÷ 4 ___ 12 ÷ 4 ___

Smaller Pile (-4 ÷ -2 = 2)

DIVIDEND	OPERATOR	DIVISOR	QUOTIENT



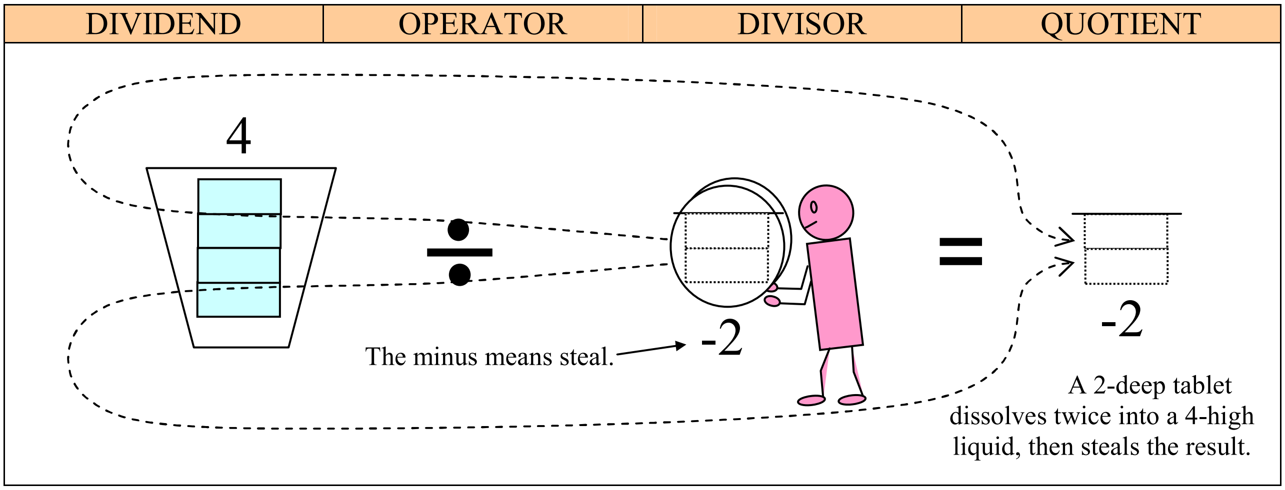
A 2-deep tablet dissolves into a 4-deep liquid 2 times.

Your turn: Dissolve the negative tablet into the negative liquid as many times as it fits.

-6 ÷ -3 ___ -6 ÷ -2 ___ -10 ÷ -2 ___

-8 ÷ -2 ___ -8 ÷ -4 ___ -12 ÷ -4 ___

Smaller Hole (4 ÷ -2 = -2)



The minus means steal.

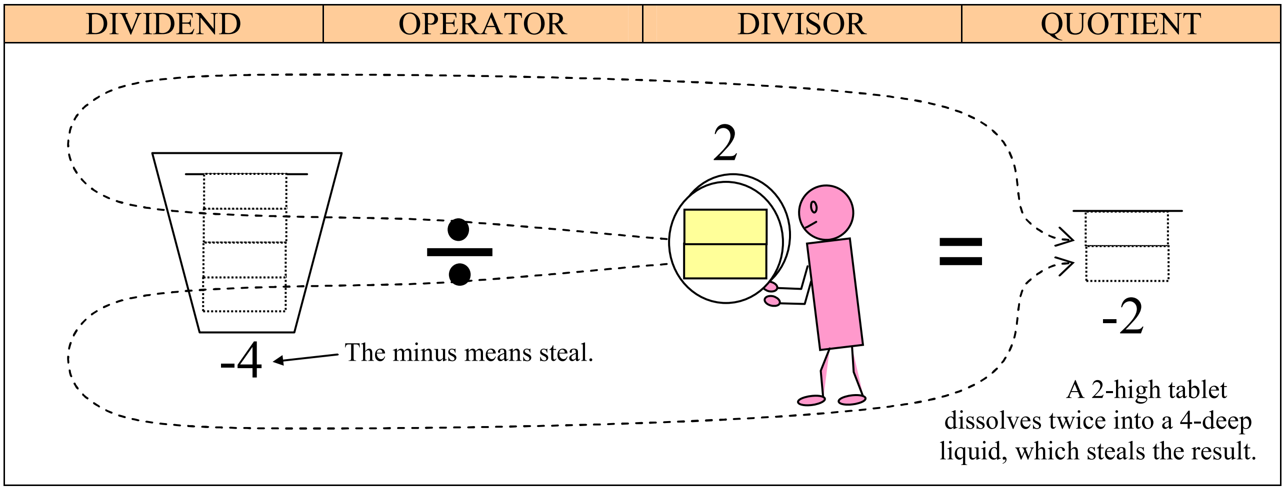
A 2-deep tablet dissolves twice into a 4-high liquid, then steals the result.

Your turn: Dissolve the negative tablet into the positive liquid, then steal the result.

6 ÷ -3 → ____   6 ÷ -2 → ____   10 ÷ -2 → ____

8 ÷ -2 → ____   8 ÷ -4 → ____   12 ÷ -4 → ____

Smaller Hole (-4 ÷ 2 = -2)



The minus means steal.

A 2-high tablet dissolves twice into a 4-deep liquid, which steals the result.

Your turn: Dissolve the positive tablet into the negative liquid, then steal the result.

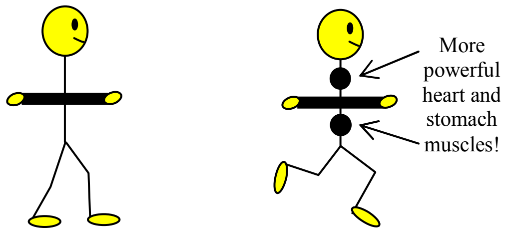
-6 ÷ 3 → ____   -6 ÷ 2 → ____   -10 ÷ 2 → ____

-8 ÷ 2 → ____   -8 ÷ 4 → ____   -12 ÷ 4 → ____

Properties of Division

Division = *Fast* Subtraction

While Subtraction takes multiple steps to deplete a number, Division does it in one step.



8–2–2–2 = 2

8 ÷ 4 = 2

Division Variations

Division has a variety of operators.

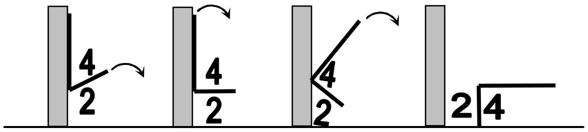
$4 \div 2$

$4/2$

$$\frac{4}{2}$$

$$2\overline{)4}$$

Imagine a hinged, rotating, detachable wall table.



Division lacks most of the common properties of Multiplication.

No Commutative Property of Division

With multiplication, you can change the number order, but *not* with division.
Example: 4 ÷ 2 = 2 is not the same as 2 ÷ 4 = 1/2.
BrainAid: Governors do *not* commute the sentences of divisive prisoners.

No Associative Property of Division

With multiplication, you can arrange numbers in any group, but *not* with division.
Example: (8 ÷ 4) ÷ 4 = 2 ÷ 4 = 1/2 is not the same as 8 ÷ (4 ÷ 4) = 8 ÷ 1 = 8
BrainAid: You should *not* associate with divisive people.

No Division Inverse

With multiplication, inverses (aka reciprocals) multiply to make 1, but *not* with division.
Example: 3 and 1/3 are inverses, but 3 ÷ 1/3 = 9. It does *not* equal one.

Division Identity Element?

With multiplication, 2 × 1 = 2 and 1 × 2 = 2. With division, this only works if 1 is the divisor.
Example: 2 ÷ 1 = 2 but 1 ÷ 2 = ½

Distributive Property of Division: A Miserly Uncle

Property: A divisor dissolves each *added* or *subtracted* item above it.

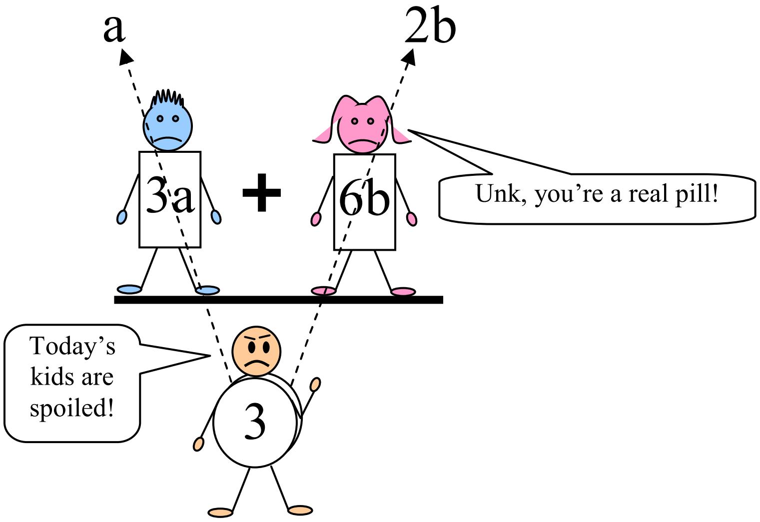
$$\frac{\overset{2a}{4a} + \overset{3b}{6b} - \overset{4c}{8c}}{2}$$

a, b, and c can be any numbers.

This is the same as the Distributive Property of Multiplication when the multiplier is a *fraction*:

$$\tfrac{1}{2}\,(4a + 6b - 8c) = \tfrac{1}{2}(4a) + \tfrac{1}{2}(6b) - \tfrac{1}{2}(8c) = 2a + 3b - 4c$$

BrainAid: Imagine a miserly uncle who decides to dissolve the wealth of his nieces and nephews. Being scrupulously *un*fair, he equally reduces whatever value each already has.



Your turn: Be a miserly uncle and equally dissolve the wealth of your nieces and nephews.

$$\frac{\overset{\underline{\quad\quad}}{6} + \overset{\underline{\quad\quad}}{9c}}{3} \qquad \frac{\overset{\underline{\quad\quad}}{8a} + \overset{\underline{\quad\quad}}{12b}}{4} \qquad \frac{\overset{\underline{\quad\quad}}{6a} + \overset{\underline{\quad\quad}}{10b} - \overset{\underline{\quad\quad}}{14c}}{2}$$

TRAP!

The Distributive Property does *not* apply
to more than the first *divided* or *multiplied* item.

$$\frac{6a \div 4b}{2} \neq 3a \div 2b$$

In cases like this, the divisor dissolves only the *first* item.

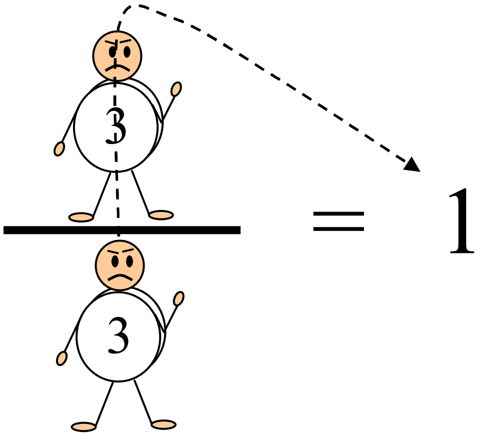
$(6a \div 4b)/2 = 3a \div 4b$

Division Property of One: A Perfect Fit

Property: Any number divided by itself equals 1.

BrainAid: A divisor dissolves (fits) exactly one time into an equal dividend.

Exception: Division by zero is *not* allowed—You can't dissolve without a tablet!



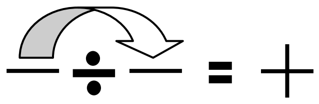
This miserly uncle
dissolves into
himself!

Division Layouts

Dividend ÷ Divisor = Quotient	$\frac{\textbf{Dividend}}{\textbf{Divisor}} = \textbf{Quotient}$
Dividend / Divisor = Quotient	$\begin{array}{r} \textbf{Quotient} \\ \textbf{Divisor}\,\overline{\big)\textbf{Dividend}} \end{array}$

Two Divided Negatives Make A Positive

When you divide two negative numbers, imagine rotating the first minus sign and placing it over the second minus sign to create a plus sign for the quotient.



Example: $\overset{\curvearrowright}{-3 \div -}1 = 3 \div +1 = +3$

TRAP!

We've seen that 2 negatives make a positive in these situations:

Subtracting a negative
3 – -1 = 3 + 1 = +4

Multiplying 2 negatives
-3 × -1 = 3 × +1 = +3

Dividing 2 negatives
-3 ÷ -1 = 3 ÷ +1 = +3

But beware!

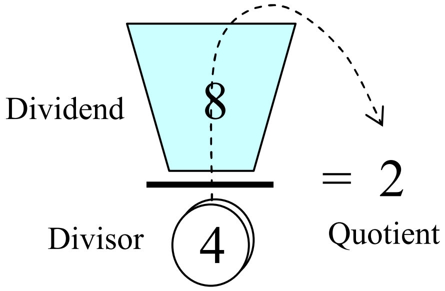
2 negatives don't always make a positive:

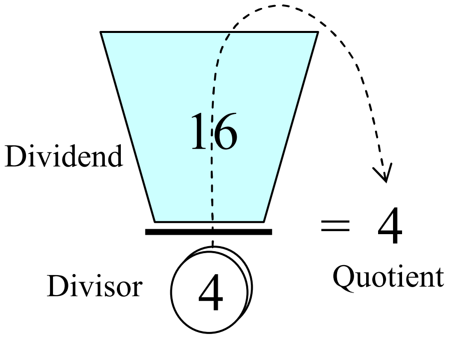
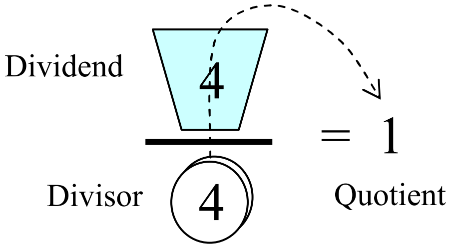
-3 + -1 = -4

-3 – 1 = -4

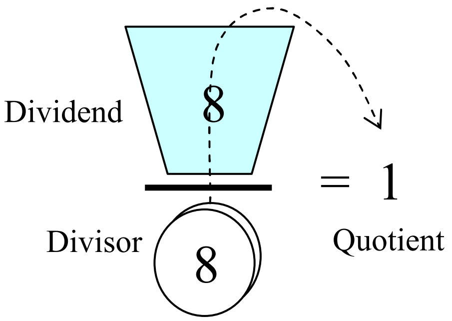
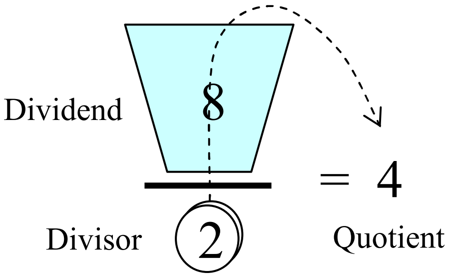
Shrink or Grow?

If a dividend or divisor increases or decreases, what happens to the quotient?
Use the mental manipulatives of a liquid-dividend and tablet-divisor to discover the relationships.



Dividend grows = Quotient grows	**Dividend shrinks = Quotient shrinks**
	

The dividend and quotient are *proportional*—they grow or shrink in the *same* direction.
BrainAid: Similar endings dividend and quotient go similarly.

Divisor grows = Quotient shrinks	**Divisor shrinks = Quotient grows**
	

The divisor and quotient are *inversely* proportional—they grow or shrink in *opposite* directions.
BrainAid: Different endings divisor and quotient go opposite.

Your turn: Fill in the blanks with "grows" or "shrinks" and the new quotient.

For each of the following, if the original division is 12/3 = 4...

...and the dividend grows to 15, the quotient _____________ to ____.

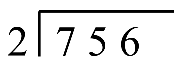
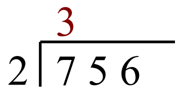
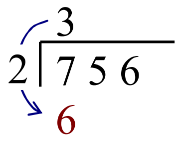
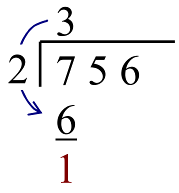
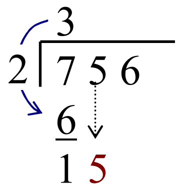
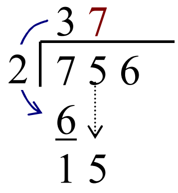
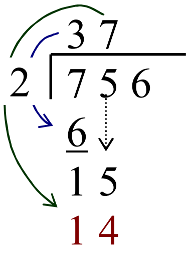
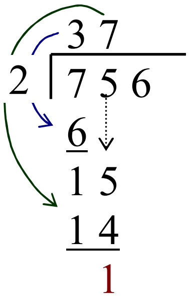
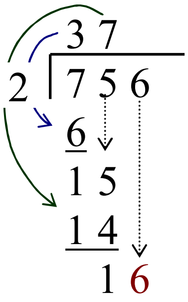
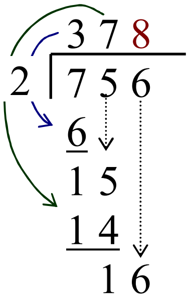
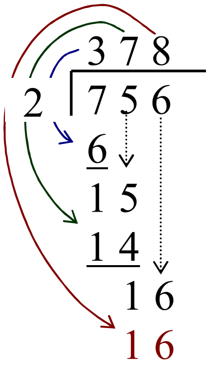
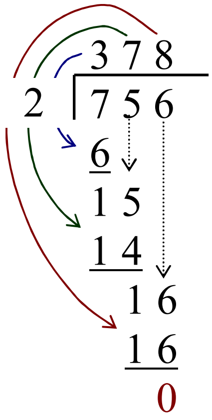
...and the dividend shrinks to 9, the quotient _____________ to ____.

...and the divisor grows to 4, the quotient _____________ to ____.

...and the divisor shrinks to 2, the quotient _____________ to ____.

Rainbow Division (aka Long Division)

Imagine that long division creates a rainbow with rain falling down.
No tricks here, just a more interesting way to visualize (and teach) long division.

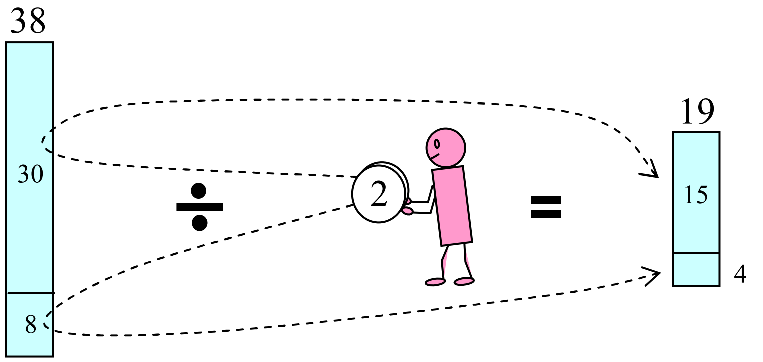
Long Division. The divisor is outside; the dividend is inside, sheltered by a roof.	Estimate the number of times the divisor will dissolve into the first digit of the dividend, and place it on top of the roof.	Multiply your estimate times the divisor, forming a rainbow to carry the *product* below the first digit of the dividend.
		
Subtract to find the *difference* between the first digits.	Due to a leaky roof, the next dividend digit falls like rain to join the *difference*.	Estimate the number of times the divisor will dissolve into the combined *difference*.
		
Multiply to create a second rainbow band and *product*.	Subtract to find the next *difference*.	Rain down the next digit of the dividend.
		
Put your estimate on the roof for the new *difference*.	Multiply to create a third rainbow band and *product*.	Subtract to find any possible remainder.
		

Mental Division (MD)

MD: Split and Halve

Tip: Mentally, it's easier to halve numbers highest-to-lowest by place values.
Trick: Split the dividend into numerical place values (100s, 10s, 1s), then start halving with the highest place value, so that the quotient is already in the order you'd think or say it.

Think
Half of 30 is 15
Half of 8 is 4
19



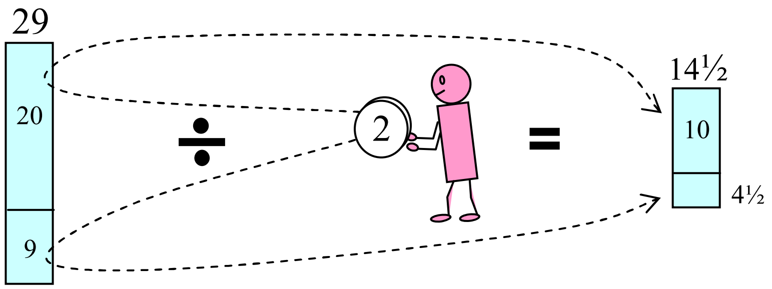
Learn The *Odd* Halves

3 ÷ 2 = 1½
5 ÷ 2 = 2½
7 ÷ 2 = 3½
9 ÷ 2 = 4½

30 ÷ 2 = 15
50 ÷ 2 = 25
70 ÷ 2 = 35
90 ÷ 2 = 45

Tip: If you forget an odd half, split the odd number into an even number + 1, then halve the even number and halve 1; e.g., 9 = 8 + 1. Half of 8 is 4, half of 1 is ½, so half of 9 is 4½.

Think
Half of 20 is 10
Half of 9 is 4½
14½



Your turn: Mentally split and halve each number.

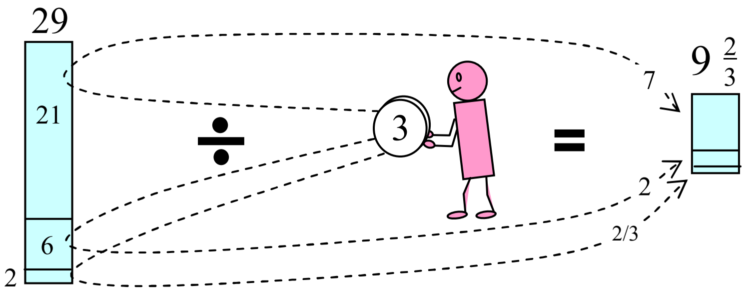
56 ÷ 2	**67 ÷ 2**	**93 ÷ 2**
50 ÷ 2 = ___	___ ÷ ___ = 30	___ ÷ ___ = ___
6 ÷ 2 = ___	___ ÷ ___ = 3½	___ ÷ ___ = ___
28	_____	_____

Bonus Tip: To divide by 4, halve the number twice.

MD: Dissolving Multiples

Tip: Mentally, it's sometimes easier to split a dividend into multiples (see p. 34) of the divisor.
Trick: Split the dividend into numbers that are divisible by the divisor,
then dissolve each number separately and join the partial quotients.

Think
29 = 21 + 6 + 2
3 into 21 is 7
3 into 6 is 2
3 into 2 is 2/3
9 2/3



It really doesn't matter which multiples you choose. The results will be the same.
Think: 29 =18 + 9 + 2; 3 into 18 is 6; 3 into 9 is 3; 3 into 2 is 2/3; 9 2/3

Your turn: Mentally split these dividends into multiples of their divisors, then dissolve.

52 ÷ 3

52 = 30 + 21 + 1
30 ÷ 3 = ___
21 ÷ 3 = ___
1 ÷ 3 = ___
______

67 ÷ 5

67 = 50 + ___ + 2
50 ÷ 5 = ___
___ ÷ 5 = ___
2 ÷ 5 = ___
______

91 ÷ 6

91 = 60 + 30 + ___
60 ÷ 6 = ___
30 ÷ 6 = ___
___÷ 6 = ___
______

MM: Factor & Dissolve

Tip: Mentally, it's sometimes easier to factor a divisor before dividing.
Trick: Factor the divisor, then divide with the smaller factors in turn.

75 ÷ 15
Think: 75 ÷ (3 × 5) = 75 ÷ 3 = 25 ÷ 5 = 5

Your turn: Factor the divisor and divide.

54 ÷ 18

54 ÷ (___ × 2) = 54 ÷ 9 = ___ ÷ 2 = ____

600 ÷ 12

600 ÷ (6 × ___) = 600 ÷ 6 = ___ ÷ 2 = ____

MD: Divide 5 = Double Tenth

Tip: Mentally, sometimes it's easier to convert a divisor of 5 into its equivalent 2 × 1/10.
Trick: To divide by 5, double the dividend (use MM: Split & Double p.36),
then divide by 10 (remove a zero or move the decimal point one left).

420 ÷ 5

Think: 2 × 420 = 840 ÷ 10 = 84

Your turn: Mentally divide by 5.

120 ÷ 5

2 × 120 = ____ ÷ 10 = ____

230 ÷ 5

2 × ____ = ____ ÷ 10 = ____

325 ÷ 5

___ × 325 = ____÷___ = ____

440 ÷ 5

___×____ = ____÷___ = ____

MD: Divide 25 = Double Double Hundredth

Tip: Mentally, sometimes it's easier to convert a divisor of 25 into its equivalent 4 × 1/100.
Trick: To divide by 25, double the dividend twice (use MM: Split & Double p. 36),
then divide by 100 (remove two zeros or move the decimal point two left).

225 ÷ 25

Think: 2 × 225 = 450 × 2 = 900 ÷ 100 = 9

Your turn: Mentally divide by 25.

350 ÷ 25

2 × 350 = _____ × 2 = _____ ÷ 100 = _____

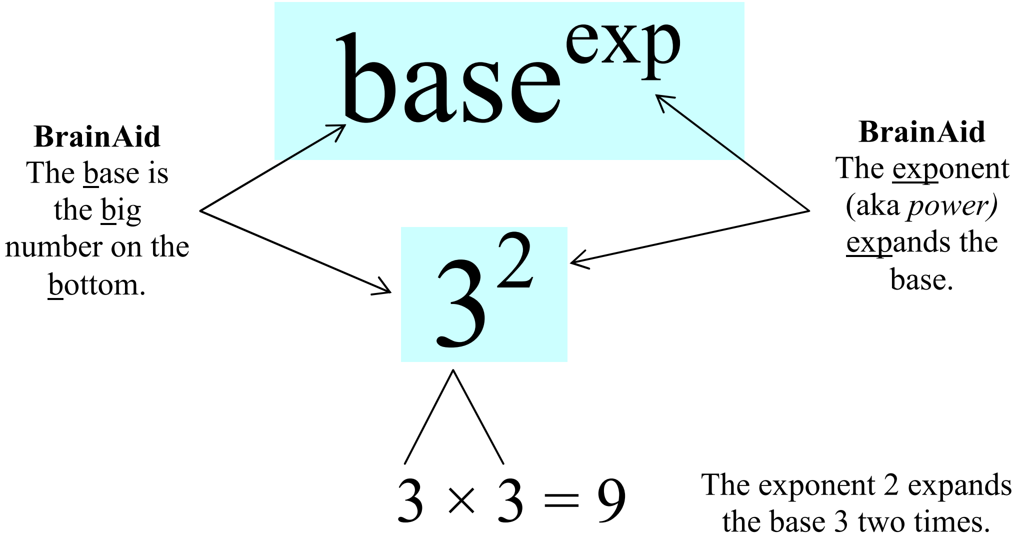
425 ÷ 25

2 × _____ = _____ × 2 = _____ ÷ 100 = _____

550 ÷ 25

2 × _____ = _____ × 2 = _____ ÷ 100 = _____

Exponentiation Expands



$$\text{base}^{\text{exp}}$$

BrainAid
The base is
the big
number on the
bottom.

BrainAid
The exponent
(aka *power)*
expands the
base.

$$3^2$$

$$3 \times 3 = 9$$

The exponent 2 expands
the base 3 two times.

Exponentiation = *Fast* Multiplication

Exponentiation is shorthand for
repeated multiplication of the same
number to produce a product.

$$2^3 = 2 \times 2 \times 2 = 8$$

Exponentiation Variations

Superscripted exponent

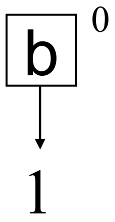
$$2^3$$

Caret [KAIR-et] symbol operator
(Used in computer formulas. The ^ is above the 6.)

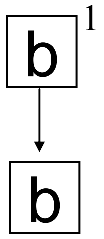
2^3

Expanding Bases: Unfolding Cards

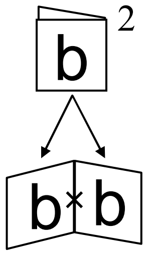
Imagine a base as a card with the letter ‘b’ on it.
The “powerful” exponent “raises” the number of base cards in a set.
Expanded cards are “hinged” together by multiplication signs.



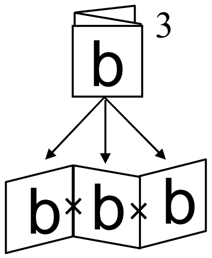
A base with
exponent 0
equals 1.
Exception: 0^0



A base raised to
the 1st power
equals the base.



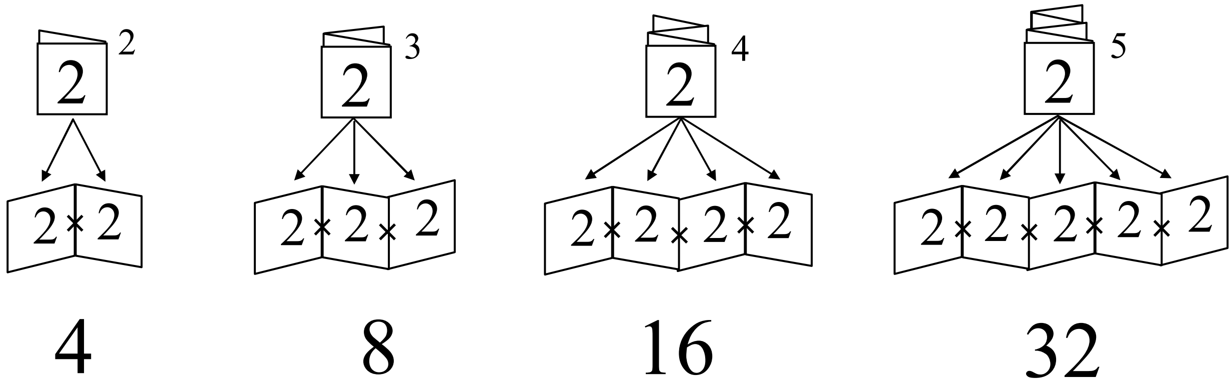
A base raised to
the 2nd power is
said to be
squared.



A base raised to
the 3rd power is
said to be
cubed.

Positive Bases: Positively Positive

Positive bases raised to any power produce positive products.



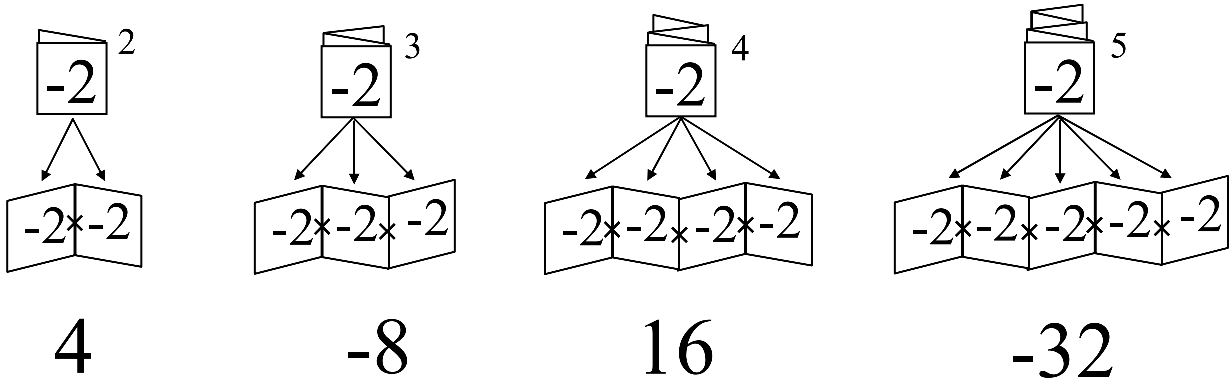
Your turn: Expand the positive base, then multiply.

$3^2 = 3 \times 3 = \_\_\_$ $4^2 = \_\_ \times \_\_ = \_\_\_$ $5^2 = \_\_ \times \_\_ = \_\_\_$

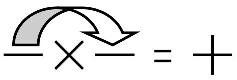
$3^3 = \_\_ \times \_\_ \times \_\_ = 27$ $4^3 = \_\_ \times \_\_ \times \_\_ = \_\_\_$ $5^3 = \_\_ \times \_\_ \times \_\_ = \_\_\_$

Negative Bases: Oddly Negative

$(–\text{base})^{\text{even}}$ = positive: Negative bases raised to even exponents produce positive products.
$(–\text{base})^{\text{odd}}$ = negative: Negative bases raised to odd exponents produce negative products.



BrainAid: Each two minus signs combine to form a plus sign.



Your turn: Expand the negative base, then multiply.

$(-3)^2 = -3 \times -3 = \_\_\_$ $(-4)^2 = \_\_ \times \_\_ = \_\_\_$ $(-5)^2 = \_\_ \times \_\_ = \_\_\_$

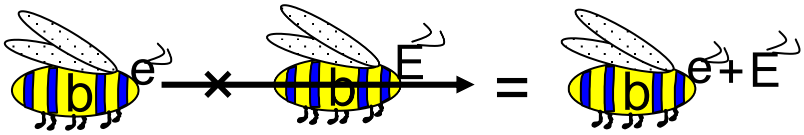
$(-3)^3 = \_\_ \times \_\_ \times \_\_ = -27$ $(-4)^3 = \_\_ \times \_\_ \times \_\_ = \_\_\_$ $(-5)^3 = \_\_ \times \_\_ \times \_\_ = \_\_\_$

Important: Negative bases must be enclosed within parentheses (see PEMDAS page 56).

Multiplying Exponents: Mad Bees / Merge Powers

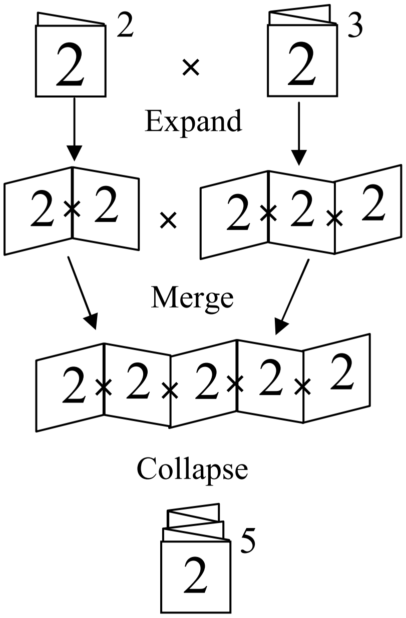
Rule 1: To multiply exponents with equal bases: merge bases and add exponents.
base^{exp} = b^{e}: Imagine that each base with its exponent is a bee.
Imagine that one bee bumps into a second bee, merging with it and making them both mad.
BrainAid: Multiply means add = Mad bees.

$$\boxed{b^{e} \times b^{E} = b^{e+E}}$$



Important: Only bees of the same breed (base) will merge into one bee.

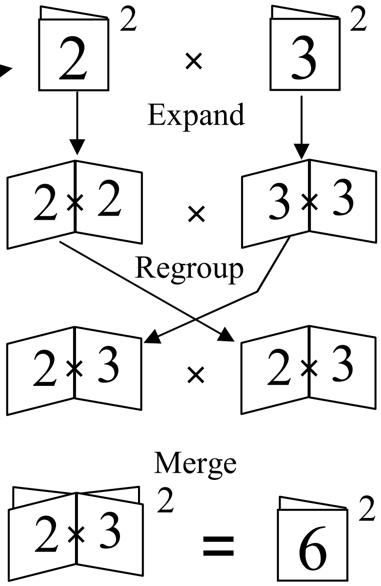
$$2^{2} \times 2^{3} = 2^{2+3} = 2^{5}$$



Rule 2: To multiply exponents with different bases but equal exponents: multiply bases and merge exponents.
BrainAid: Bees of different breeds (bases) merge same powers.

$$\boxed{b^{e} \times B^{e} = bB^{e}}$$

$$2^{2} \times 3^{2} = 6^{2}$$



Square Roots = b^{½}

$2^{½} \times 8^{½}$

$(2 \times 8)^{½}$

$16^{½}$

$\sqrt{2} \times \sqrt{8}$

$\sqrt{2 \times 8}$

$\sqrt{16}$

TRAP!

You can *not* merge different bases with different exponents.

$$2^{2} \times 3^{3} \neq (2 \times 3)^{2+3} \neq 6^{5}$$

You *can* expand each base, then multiply.

$$2^{2} \times 3^{3} = 2 \times 2 \times 3 \times 3 \times 3 = 108$$

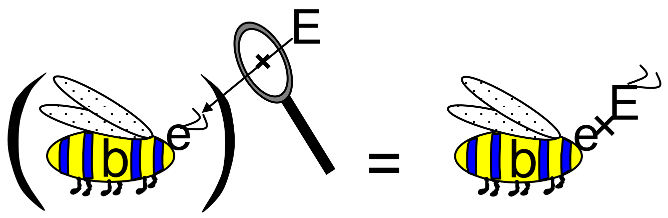
Your turn: Multiply using Rule 1 or Rule 2 accordingly.

$2^{3} \times 2^{4} =$ ____ $3^{5} \times 3^{3} =$ ____ $2^{3} \times 3^{3} =$ ____ $4^{2} \times 5^{2} =$ ____

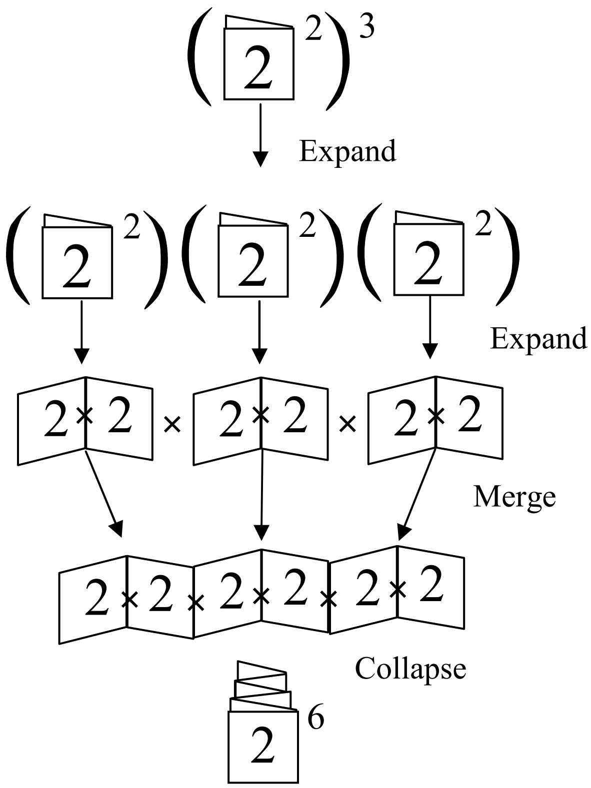
Raising a Base: Ram Bee / Ram Bees

Rule 1: To raise a base^{exp} to an external power, multiply the exponents.
Imagine a magnify glass using its power to ram the exponents together.
BrainAid: Raise means multiply = Ram bee.

$$\boxed{(b^e)^E = b^{e\times E}}$$



$$(2^2)^3 = 2^{2\times3} = 2^6$$



Rule 2: To raise unlike bases to an external power, distribute the external exponent over the inner ones.
BrainAid: Ram multiple bees.

$$\boxed{(b^eB^E)^{\mathbf{E}} = b^{e\times\mathbf{E}}B^{E\times\mathbf{E}}}$$

$$(2^2 3^3)^2 = 2^{2\times2}\,3^{3\times2} = 2^4 3^6$$

TRAP!

$$(2^2+3^3)^2 \neq 2^4+3^6$$

$$(2^2+3^3)^2 = (2^2+3^3)(2^2+3^3)$$

Your turn: Raise the base/s by multiplying the exponents.

$(2^3)^4 =$ ____ $(3^3)^3 =$ ____ $(3^3 4^2)^2 =$ ____ $(4^4 5^3)^2 =$ ____

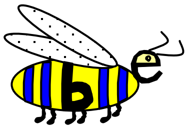
Negative Exponents: Screening Bees

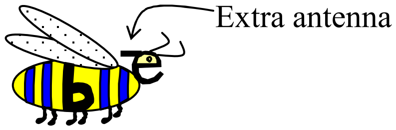
Inverting the base$^{\text{exp}}$ reverses the sign of the exponent.

Using vertical division, inverting means to move from top to bottom or bottom to top.

BrainAid: Imagine the ‘b’ in base is the body of a bee, and the ‘e’ in exponent is the head of a bee. Imagine that a bee with a negative exponent has an extra antenna at the back of its head.

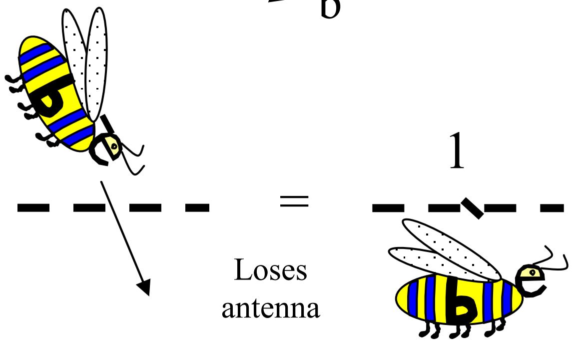
$$\text{base}^{\text{exp}} = b^{e} \qquad \text{base}^{-\text{exp}} = b^{-e}$$



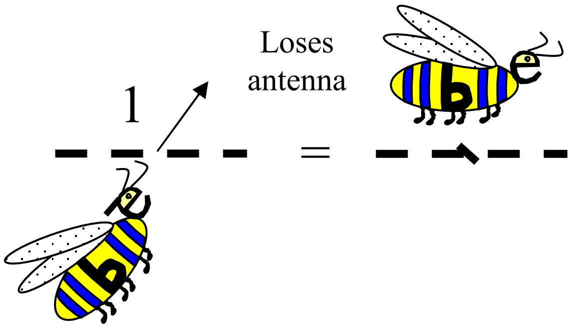


BrainAid: When a negative-exponent bee flies down or up through a screen (division line), its extra antenna gets caught and breaks off; i.e., the negative exponent becomes positive.

$$b^{-e} = \frac{1}{b^{e}}$$

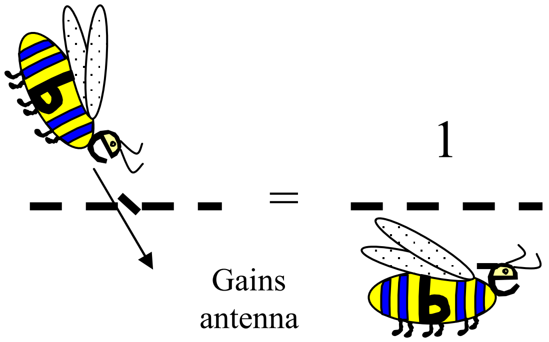


$$\frac{1}{b^{-e}} = b^{e}$$

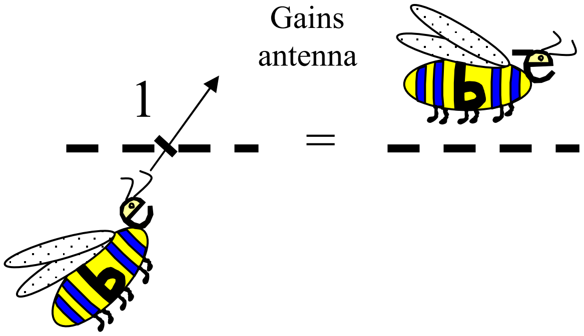


BrainAid: When a positive-exponent bee flies down or up through a screen (division line), it gains the extra antenna left by a negative-exponent bee; i.e., the positive exponent becomes negative.

$$b^{e} = \frac{1}{b^{-e}}$$



$$\frac{1}{b^{e}} = b^{-e}$$



Your turn: Invert the base and reverse the exponent sign.

$2^{-3} = \frac{1}{\quad}$ $\qquad$ $\frac{1}{2^{-3}} =$ $\qquad$ $4^{5} = \frac{1}{\quad}$ $\qquad$ $\frac{1}{4^{5}} =$

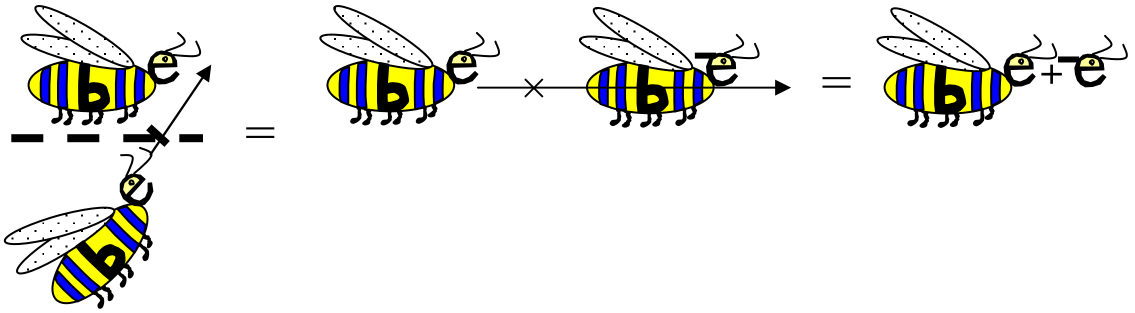
Dividing Bases: Screening Mad Bees

To divide bases that are alike, invert one base, then multiply.

BrainAid: Imagine two bees separated by a screen. As one flies up or down though the screen, it gains or loses an antenna before it joins the other. Then they're multiplied using merge and add = mad.

Flies up / Gains antenna

$$\frac{b^e}{b^e}\nearrow = b^e \times b^{-e} = b^{e + -e}$$



Important: Only bees of the same breed (base) will merge into one bee.

Flies up / Loses antenna

$$\frac{b^e}{b^{-e}}\nearrow = b^e \times b^e = b^{e+e}$$

Flies down / Loses antenna

$$\frac{b^{-e}}{b^e}\searrow = \frac{1}{b^e \times b^e} = \frac{1}{b^{e+e}}$$

Flies up / Gains antenna

$$\frac{b^{-e}}{b^e}\nearrow = b^{-e} \times b^{-e} = b^{-e + -e}$$

Flies down / Gains antenna

$$\frac{b^e}{b^{-e}}\searrow = \frac{1}{b^{-e} \times b^{-e}} = \frac{1}{b^{-e + -e}}$$

Your turn: Divide by inverting the indicated base, then multiplying.

$$\frac{2^5}{2^3}\nearrow = \qquad \frac{2^5}{2^{-3}}\nearrow = \qquad \frac{2^{-5}}{2^3}\nearrow =$$

$$\frac{2^5}{2^3}\searrow = \frac{1}{\qquad} \qquad \frac{2^5}{2^{-3}}\searrow = \frac{1}{\qquad} \qquad \frac{2^{-5}}{2^3}\searrow = \frac{1}{\qquad}$$

PEMDAS Prioritizes

Priority Of Operators

When a math problem includes multiple operations, for example, addition and division and subtraction, how do you know which one to start with? It might seem reasonable to start on the left and proceed to the right, but this isn't always the case.

To avoid confusion mathematicians have assigned priorities to each operator. That is, certain operations must be done *before* others, no matter where they appear in the problem. If you don't follow these priorities precisely, you'll get the wrong answer.

In order, the priorities are: Parentheses, Exponentiation, Multiplication, Division, Addition, Subtraction. Two traditional memory hints are commonly used to teach these priorities:

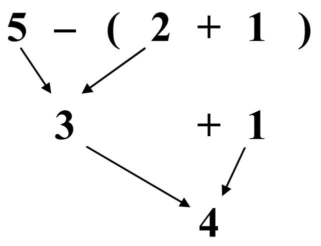
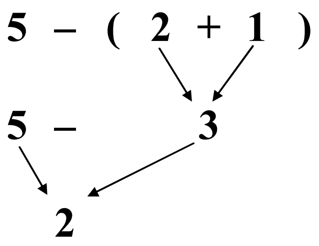
- **Acronym:** PEMDAS [PEM-dass]
- **Acrostic:** Please Excuse My Dear Aunt Sally

Let's examine each operation in order.

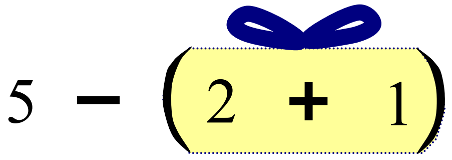
Parentheses
have Priority
Exponentiation
Expands
Multiplication
Magnifies
Division
Dissolves
Addition
Attaches
Subtraction
Steals

Parentheses: Open Me First!

An operation inside a set of parentheses has priority over an operation outside.

Incorrect Priority	Correct Priority
	

BrainAid: Imagine parentheses as a package which says, "Open me *first!*"



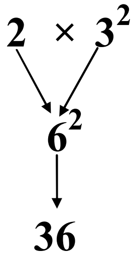
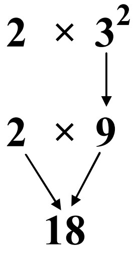
If a problem has multiple sets of (parentheses) or [brackets] or {braces}, work from the inside out.

(third (second (first))) or {third [second (first)] }

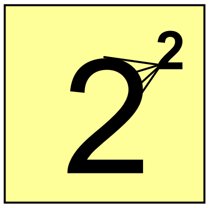
Example: 2 (3 × (4 + 5)) = 2 (3 × 9) = 2 (27) = 54

Exponentiation: A Higher Power

Raising a number to a power has priority over all operations outside of parentheses.

Incorrect Priority	Correct Priority
	

BrainAid: Imagine lines of force emanating from the 'higher power' exponent down towards its base. The exponent exerts a powerful influence, putting pressure on the base to expand before it gets involved with any other operations.



Exponentiation Negation Controversy

Negation reverses the sign of a number.
Essentially, it's like subtraction but with no minuend.
Example: 2-1 is a subtraction; -1 is a negation, as is - -1.
Regarding priority order, negation is treated like subtraction.

THE PROBLEM

Computer spreadsheet programs and some calculators
handle exponentiation and negation in nonstandard ways.

MATHEMATICALLY

$$-2^2 = -(2 \times 2) = -4$$

SPREADSHEETS / SOME CALCULATORS

$$-2^2 = (-2) \times (-2) = +4$$

THE SOLUTION

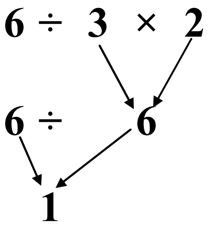
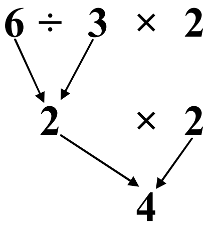
Before creating formulas that use exponents,
be sure to test how your computer/calculator handles them.
If possible, use parentheses to force the priority you need.

$$-(2^2) = -(2 \times 2) = -4$$

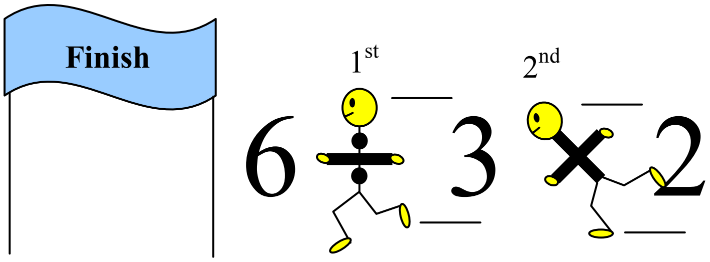
$$(-2)^2 = (-2) \times (-2) = +4$$

Multiplication or Division: Fast Runners

These operators have left-to-right priority; i.e., first come, first done.

<table>
<tr><th>Incorrect Priority</th><th>Correct Priority</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

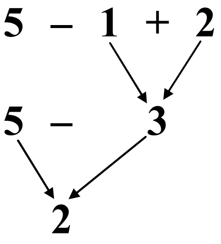
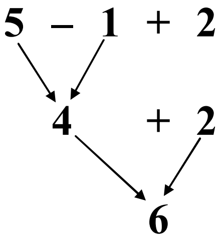
BrainAid: Imagine that multiplication (aka *fast* addition) and division (aka *fast* subtraction) are runners in a race. Whoever is closest to the finish line (on the left) wins priority over the other.



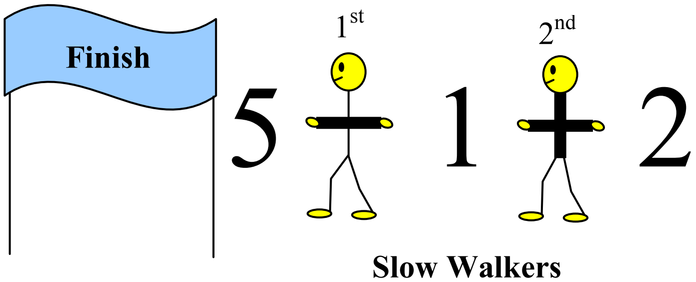
Fast Runners

Addition or Subtraction: Slow Walkers

These operators have the lowest priority and are also done in left-to-right order.

<table>
<tr><th>Incorrect Priority</th><th>Correct Priority</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

BrainAid: Imagine that the addition and subtraction operators can only walk instead of run the race. Whoever is closest to the finish line has priority over the other. Of course, any multiplication or division runners will finish ahead of the walkers and have higher priority.



Slow Walkers

Multiple Operator Problems: Give me a Number, please!

When you encounter multiple operator problems, follow these steps.

You encounter a problem with four operators.

$$12 - 5 \times (2 + 4) \div 3$$

Using PEMDAS rules, assign priority numbers above each operator.

This will ensure you follow the correct order of operations.
Circle each priority number, so you don't confuse it with a number being operated on.

$$12 \overset{④}{-} 5 \overset{②}{\times} (2 \overset{①}{+} 4) \overset{③}{\div} 3$$

Draw thin vertical guidelines between each number and operator.

This will keep things organized and ensure you don't overlook a number or operator.

$$12 \overset{④}{-} 5 \overset{②}{\times} (2 \overset{①}{+} 4) \overset{③}{\div} 3$$

Calculate with the first priority operator.

Place the results directly *beneath* the operator and draw arrows to the result.
Cross out the first priority number.

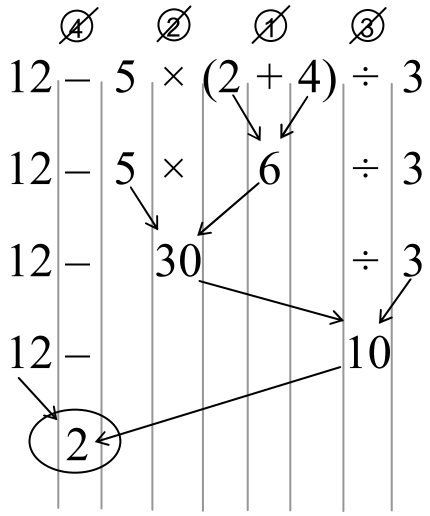
$$12 \overset{④}{-} 5 \overset{②}{\times} \underbrace{(2 \overset{\cancel{①}}{+} 4)}_{6} \overset{③}{\div} 3$$

Carry down and rewrite the remaining numbers and operators in their respective columns.

$$\begin{array}{ccccccccc} 12 & \overset{④}{-} & 5 & \overset{②}{\times} & (2 & \overset{\cancel{①}}{+} & 4) & \overset{③}{\div} & 3 \\ & & & & \searrow & & \swarrow & & \\ 12 & - & 5 & \times & & 6 & & \div & 3 \end{array}$$

(continued on next page)

Repeat the process for each remaining operator.
Cross out each priority number *after* you perform its operation (not before).
Circle the final answer.



Algorithms

This procedure may seem cumbersome. But it’s very easy to make errors with multi-operator problems. In general, it’s a good idea to follow a set procedure, aka algorithm [AL-goh-RI-thhum], when working any math problems. It’s too easy to have your mind wander or get distracted, especially on longer, more involved problems. An algorithm will help to keep you on track until you reach the correct solution. It can also reduce the mental effort required if you have a pattern to follow.

Twice Done is Well Done

Benjamin Franklin may not have been thinking of math problems when he coined this phrase, but it certainly applies. If you have time, always try to solve a math problem twice and in two different ways. If you get the same answer both times, you’ve likely done it correctly. Doing a problem a second time, and in a different way if possible, increases the likelihood of uncovering any mistakes in your logic or calculations.

Your turn: Using PEMDAS rules, assign priority numbers and solve in order.

$5 + 2 \times 3$

$12 - 8 \div 4$

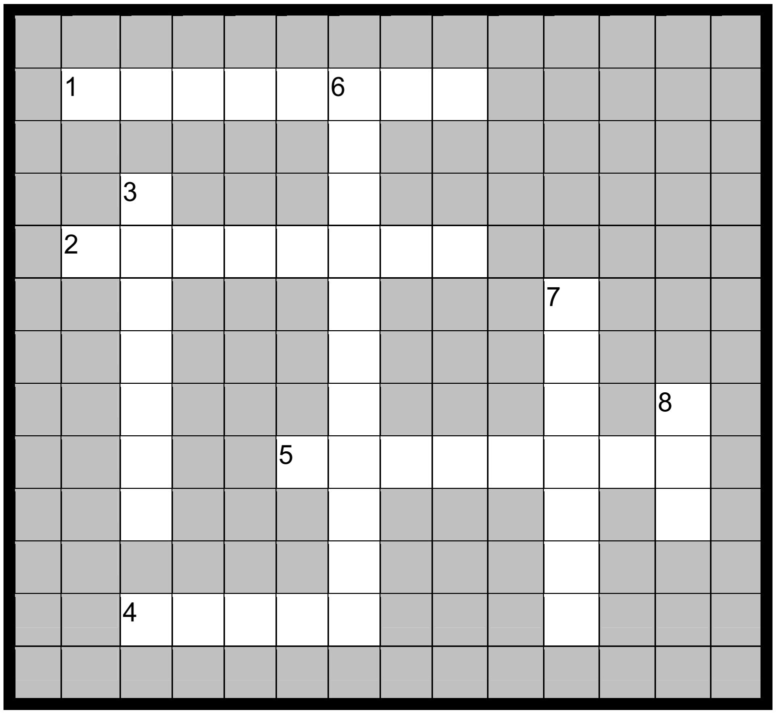
$12 - 4 \times 3 + 2$

$12 - 9 \div 3 \times 2$

$8 + 4 \div 2^2 - 1$

$2 \times (4 + 2)^2 \div 8 - 4$

BrainDrain #3



Fill in the Crossword Puzzle

Across

1. To _____ bases, add their exponents.
2. Inverting a base changes the sign of its _____.
4. If a divisor shrinks, the quotient _____.
5. _____ reverses the sign of a number.

Down

3. Exponentiation _____ the base.
6. _____ have the highest priority.
7. If a dividend shrinks, the quotient _____.
8. 3/3 = 1 demonstrates the Division Property of _____.

True/False

Write T or F in the blanks.

1____ The distributive property does *not* hold for division.

2____ Two negatives *always* make a positive.

3____ Only equal bases can be merged when multiplied.

4____ Negation and subtraction are the same thing.

5____ To raise a base to a power, add the exponents.

Too Many Options?

With multiple ways to solve most mental math problems, you might find yourself bouncing from one technique to another without following through to an answer. If so, relax, pick one approach, even if it's not the most efficient, and get an answer. Over time, you'll tend to favor certain techniques over others.

Must I Always Use Mental Manipulatives?

As you practice mental math techniques in your daily life, you'll get better at solving problems directly with symbols, without picturing piles, holes, magnifying glasses, tablets, buckets, cards, bees, or other mental manipulatives. This is okay, because the goal is to get the correct answer by whatever means. However, when you're distracted and not able to fully focus, it often helps to slow down and visualize mental manipulatives to help you concentrate.

Answer Key

Addition Attaches

Page 8: Larger Pile (3 + 1 = 4)
Top Row: 5, 7, 8; Bottom Row: 9, 6, 10

Page 9: Larger Hole (-3 + -1 = -4)
Top Row: -5, -7, -8; Bottom Row: -9, -6, -10

Page 10: Smaller Pile (3 + -1 = 2)
Top Row: 1, 3, 4; Bottom Row: 5, 2, 8

Page 11: Smaller Hole (-3 + 1 = -2)
Top Row: -1, -3, -4; Bottom Row: -5, -2, -8

Page 14: MA: Borrow
Top Row: 7 | 50, 3, 53; Bottom Row: 40, 41 | 5, 80, 85. *More Borrowing:* Top Row: 143, 185, 1046; Bottom Row: 198, 170, 1315

Page 15: MA: Find 10s
Top: 9, 19; Middle: 10, 10, 20; Bottom: 10, 5, 15

Page 15: MA: Stack Signs
Left:-6, 7, -2, Center: 2, -5, 8, -6, 2;
Right: 2, 1, -1, 6,-5, 1

Page 16: MA: Split & Join
Left Column: 80, 6; 86 | 50, 90, 3, 5, 95; Right Column: 70, 15, 5, 85 | 90, 70, 6, 7, 160, 10, 173

BrainDrain #1

Page 17
Crossword Puzzle: Across: 1. commutative, 3. highest, 5. opposite, 7. operator, 8. Identity; Down: 2. zero, 4. more, 6. symbol, 9. negative
True/False: 1T, 2F (whole numbers have no negatives), 3T, 4T, 5F (zero = Identity). 6F (ordinal)

Subtraction Steals

Page 18: Smaller Pile (3 – 1 = 2)
Top Row: 1, 3, 3; Bottom Row: 5, 6, 3

Page 19: Smaller Hole (-3 – -1 = -2)
Top Row: -1, -3, -4; Bottom Row: -3, -6, -2

Page 20: Larger Pile (3 – -1 = 4)
Top Row: 7, 6, 9; Bottom Row: 11, 12, 15

Page 21: Larger Hole (-3 – 1 = -4)
Top Row: -6, -7, -8; Bottom Row: -10, -12, -14

Page 23: MS: Bump
Bump Up: Top Row: 10, | 43, 13; Bottom Row: 77, 40, 37, | 76, 40, 36 (bump up 2)
Bump Down: Top Row: 10, | 49, 19; Bottom Row: 69, 40, 29 | 68, 40, 28 (bump down 2)

Page 24: MS: Split & Steal
Left Column: 40, 2 | 40, 20, 8, 5, 23;
Right Column: 40, -2, 38 | 40, 20, 8, 9, 19

Page 25: MS: Dig Pile
Left: 20, 4, 8, 16; Right: 100, 0, 50, 60, 90, 5, 85

Page 26: MS: Fill Up
Top Row: 63, 142, 68;
Bottom Row: 626, 1418, 671

Page 27: MS: Span & Join
Top Row: 50, 76; Bottom Row: 75, 30
Century Span: 2000, 64, 2, 66

Multiplication Magnifies

Page 28: Larger Pile (2 × 3 = 6)
Top Row: 8, 9, 12; Bottom Row: 15, 14, 9

Page 28: Larger Pile (-2 × -3 = 6)
Top Row: 10, 12, 8; Bottom Row: 20, 14, 9

Page 29: Larger Hole (2 × -3 = -6)
Top Row: -8, -6, -12; Bottom Row: -15, -14, -9

Page 29: Larger Hole (-2 × 3 = -6)
Top Row: -10, -12, -6; Bottom Row: -20, -16, -9

Page 31: Distributive Property Multiplication
Top Row: 6b | 24a, 20; Bottom Row: 4 | 3a, 5b

Page 32: Multiplicative Inverse
Top Row: ¼ , -½, 5; Bottom Row: 1/5, -1/7, 6/5

Page 34: Multiple Table
Top Row: 6, 10; Middle Row: 6, 12, 18;
Bottom Row: 12, 20

Page 35: Factoring Tricks
4 Yes, because the last two digits are a multiple of 4 (80/4 = 20).
5 Yes, because 5580 ends in 0.
6 Yes, because 5580 fits the tricks for both 2 & 3 (i.e., even and a multiple of 3).
9 Yes, because 5+5+8+0 = 18, which is a multiple of 9 (18/9 = 2).

Page 35: Factor Trees
3 × 5 | 2 × 2 × 2 × 2 | 2 × 3 × 3

Page 36: MM: Split & Double
Left: 60, 8, 68; Center: 40, 80, 7, 14, 94;
Right: 70, 140, 8, 16, 156

Page 36: MM: Split & Magnify
Left: 90, 12, 102; Center: 40, 240, 7, 42, 282; Right: 60, 420, 5, 35, 455

Page 37: MM: Factor & Magnify
Left: 2, 30, 180; Right: 8, 90, 720

Page 37: MM: Multiply 5 = Half Ten
Top Row: 12, 120 | 68, 34, 340
Bottom Row: 140, 70, 700 | 244, 122, 1220

Page 37: MM: Multiply 25 = Quarter Hundred
Top Row: 6, 600 | 36, 9, 900
Bottom Row: 88, 22, 2200 | 320, 80, 8000

Page 38: MM: 11 Split & Insert
Left: 6 (6+3) 3, 693;
Right: 7 (7+9) 9, 7 (16) 9, 869

Page 38: MM: 5-End Squared
Left: 1225; Right: 2025

BrainDrain #2

Page 39
Crossword Puzzle: Across: 1. multiplier, 2. distributive, 5. one, 6. noun; Down: 3. product, 4. associative, 7. zero, 8. prime, 9. reciprocal
True/False: 1F, 2F, 3F, 4T, 5F (57=3×19)

Division Dissolves

Page 40: Smaller Pile (4 ÷ 2 = 2)
Top Row: 2, 3, 5; Bottom Row: 4, 2, 3

Page 40: Smaller Pile (-4 ÷ -2 = 2)
Top Row: 2, 3, 5; Bottom Row: 4, 2, 3

Page 41: Smaller Hole (4 ÷ -2 = 2)
Top Row: -2, -3, -5; Bottom Row: -4, -2, -3

Page 41: Smaller Hole (-4 ÷ 2 = 2)
Top Row: -2, -3, -5; Bottom Row: -4, -2, -3

Page 43: Distributive Property of Division
Left: 2, 3c; Center: 2a, 3b; Right: 3a, 5b, 7c

Page 45: Shrink or Grow?
From Top: grows 5, shrinks 3, shrinks 3, grows 6

Page 47: MD: Split & Halve
Left: 25, 3; Center: 60, 2, 7, 2, 33½;
Right: 90, 2, 45, 3, 2, 1½, 46½

Page 47: MD: Dissolving Multiples
Left: 10, 7, 1/3, 17 1/3; Center: 15, 10, 15, 3, 2/5, 13 2/5; Right: 1, 10, 5, 1, 1/6, 15 1/6

Page 48: MM: Factor & Dissolve
Top: 9, 6, 3; Bottom: 2, 100, 50

Page 49: MD: Divide 5 = Double Tenth
Top Row: 240, 24 | 230, 460, 46
Bottom Row: 2, 650, 10, 65 | 2, 440, 880, 10, 88

Page 49: MD: Divide 25 = Double Double Hundredth
Top: 700, 1400, 14; Center: 425, 850, 1700, 17;
Bottom: 550, 1100, 2200, 22

Exponentiation Expands

Page 51: Positive Bases
Top: 9 | 4, 4, 16 | 5, 5, 25
Bottom: 3, 3, 3 | 4, 4, 4, 64 | 5, 5, 5, 125

Page 51: Negative Bases
Top: 9 | -4, -4, 16 | -5, -5, 25
Bottom: -3, -3, -3 | -4, -4, -4, -64 | -5, -5, -5, -125

Page 52: Multiplying Exponents
2^7, 3^8, 6^3, 20^2

Page 53: Raising a Base
2^{12}, 3^9, $3^6 4^4$, $4^8 5^6$

Page 54: Negative Exponents
$1/2^3$, 2^3, $1/4^{-5}$, 4^{-5}

Page 55: Dividing Bases
Top Row: 2^2, 2^8, 2^{-8};
Bottom Row: $1/2^{-2}$, $1/2^{-8}$, $1/2^8$

PEMDAS Prioritizes

Page 61
Top Row: 2, 1; 5+6; 11 | 2, 1; 12-2; 10
Middle Row: 2, 1, 3; 12-12+2; 0+2; 2 | 3, 1, 2; 12-3×2; 12-6; 6
Bottom Row: 3, 2, 1, 4; 8+4÷4-1; 8+1-1; 9-1; 8 | 3, 1, 2, 4, 5; 2×6^2÷8-4; 2×36÷8-4; 72÷8-4; 9-4; 5

BrainDrain #3

Page 62
Crossword Puzzle: Across: 1. multiply, 2. exponent, 4. grows, 5. negation; Down: 3. expands, 6. parentheses, 7. shrinks, 8. one
True/False: 1F, 2F, 3T, 4F, 5F

Now try my next two books:

Fraction Fun
Algebra Antics